ON PROJECTIVE, INJECTIVE AND FLAT SEMIMODULES

BY

RANGGA GANZAR NOEGRAHA

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Dissertation Committee

Dr. Jawad Abuihlail (Adviser)

Dr. Abdallah Laradji (Member)

Salah-Eddine Kabbaj

Dr. Salah-Eddine Kabbaj (Member)

Dr. Othman Echi (Member)

Dr. Abdeslam Mimouni (Member)

Dr. Hussain S. Al-Attas
Department Chairman

Dr. Salam A. Zummo
Dean of Graduate Studies

23/5/18

Date

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To my parents,

my wife,

and my children,

thank you for everything.

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I dedicate this dissertation to my family. Thank you for your unconditional and continuous support.

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Notation

$\leq_S$	S-subsemimodule
$\leq_S^{\oplus}$	Direct summand
$\simeq$	Isomorphic
$\equiv_N$	The Bourne relation
$\oplus$	The direct summand
$\otimes_S$	The tensor product
$\lfloor \cdot \rfloor$	The floor function
$[m]_\rho$	The congruence class of m
Δ_M	The diagonal relation
0_S	The additive identity
1_S	The multiplicative identity
ACC (DCC)	The ascending (descending) chain condition
$\mathbb{B}$	$\{0, \infty\}$
$B(n, i)$	The Alarcon & Anderson's semiring
$coker(f)$	The cokernel map
$Coker(f)$	The cokernel set
$\mathbf{E}_M$	The endomorphism semiring
$Hom_S(M, N)$	The set of S-linear maps from M to N
$I^+(S)$	The set of additively idempotent elements

$I^{\times}(S)$	The set of multiplicatively idempotent elements
$I(S)$	The set of idempotent elements
id_M	The identity map
$K^+(S)$	The set of cancellative elements
$ker(f)$	The kernel map
$Ker(f)$	The kernel set
$\overline{L}$	Subtractive closure
$\varinjlim$	The direct limit
$\varprojlim$	The direct colimit
M/ρ	The quotient semimodule over congruence relation ρ
M/N	The quotient semimodule over N
$M_n(S)$	The set of $n \times n$ matrices
$\mathbb{Q}^+$	The set of non-negative rational numbers
$\mathbb{R}^+$	The set of non-negative real numbers
$S^{(\Lambda)}$	The direct sum of S
S^{Λ}	The direct product of S
SM	The category of S-semimodules
$Span\{\cdot\}$	Generated semimodule
$V(S)$	The set of elements which have additive inverse
$\mathbb{Z}^+$	The set of non-negative integers
$Z(S)$	Zeroid

THESIS ABSTRACT

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Projective, injective and flat modules play an important role in the study of the category of modules over rings and in the characterization of various classes of rings. Several characterizations of projective (resp., injective, flat) objects which are equivalent for modules over rings are not necessarily equivalent for semimodules over an arbitrary semiring. We study several of these notions, in particular the *e-projective, e-injective* and *e-flat semimodules*, introduced recently by Abuhlail using his new notion of *exact sequences* of semimodules. We also investigate possible characterizations of special classes of semirings (e.g., von Neumann regular, Noetherian, Artinian and semisimple semirings) using special classes of their projective, injective and flat semimodules.

الملخص

تلعب الحلقيات الإسقاطية والمتباينة والتآلفية دورا مهما في دراسة فئة الحلقيات على الحلقات وتوصيف العديد من الصفوف من الحلقات. العديد من التوصيفات للأشياء الإسقاطية (على التوالي: المتباينة والتآلفية) التي تتكافأ في فئة الحلقيات على الحلقات تصبح غير متكافئة عندما يدور الحديث حول أنصاف الحلقيات على أنصاف الحلقات. ندرس العديد من تلك المفاهيم، بخاصة أنصاف الحلقيات الـe-إسقاطية والـe-متباينة والـe-تآلفية التي قدمها أبو هليل باستخدام مفهوم جديد للمتتاليات التامة لأنصاف الحلقيات. نبحث أيضا توصيفات محتملة لصفوف خاصة من أنصاف الحلقات (مثلا: أنصاف حلقات فون نويمان المنتظمة، وأنصاف الحلقات النوثيرية، وأنصاف الحلقات الآرتينية، وأنصاف الحلقات نصف البسيطة) باستخدام صفوف خاصة من حلقياتها الإسقاطية والمتباينة والتآلفية.

Introduction

The importance of *semirings* (defined, roughly, as rings not necessarily with subtraction) stems from the fact that they can be considered as a generalization of both rings and distributive bounded lattices. Moreover, semirings, and their *semimodules* (defined, roughly, as modules not necessarily with subtraction), proved to have wide applications in many aspects of Computer Science and Mathematics, e.g., Automata Theory [21], Tropical Geometry [17] and Idempotent Analysis [32]. Many of these applications can be found in Golan's book [18], which is considered a main reference in this topic.

A systematic study of semimodules over semirings was carried out by M. Takahashi in a series of papers 1981-1990. However, he defined two main notions, namely *tensor products* [38] and *exact sequences* [37], as defined in the category of modules over rings. Unfortunately, his approach did not take into consideration the *significant* difference between the nice *Abelian* nature of the category of modules over a ring and the subtle *non-additive* nature of the category of semimodules over an arbitrary semiring. Nevertheless, his definitions were used by most of the researchers in this topic in the 20th century.

By the beginning of the 21st century, several researchers began to use a more natural notion of tensor products of semimodules ([27]) with which the category of semimodules over a commutative semiring is *monoidal* rather than *semimonoidal* [2]. On the other hand, several notions of exact sequences were introduced ([33]), each of which with advantages and disadvantages. One of the most recent notions is due to Abuhlail [1] and is based on an intensive study of the nature of the category of semimodules over a semiring.

Several papers by Abuhlail, I'llin, Katsov and Nam (among others) prepared the stage for a homological characterization of special classes of semirings using special classes of projective, injective and flat semimodules (cf. [30], [22], [26], [1], [31], [5], [25], [6]).

The notions of projective and injective (as well as flat) objects can be defined in any category (with filtered colimits) relative to a suitable *factorization system* of its arrows. Projective, injective and flat semimodules have been studied intensively (see [17] for details). Recently, left (right) V-semirings, all of whose *congruence-simple* left (right) semimodules are injective have been completely characterized in [5], and *ideal-semisimple* semirings all of whose left cyclic semimodules are projective have been investigated in [25].

In addition to the *categorical notions* of *projective*, *injective* and *flat semimodules* over a semiring, several other notions were considered in the literature, e.g., the so called k-projective *semimodules* [12], i-*injective semimodules* [11] and *mono-flat semimodules* [27]. One reason for the interest of such notions is the phenomenon that assuming that *all* semimodules of a given semiring S are projective or injective (in the categorical sense) forces the semiring to be a *ring* ([22, Theorem 3.4]). Moreover, a commutative semiring all of whose semimodules are flat is a von Neumann regular *ring* [28, Theorem 2.11].

Using the *monoidal* tensor product of semimodules and a new notion of exact sequences of semimodules over a semiring, Abuhlail introduced ([3], [4]) the *homological notions* of *exactly projective* (resp., *exactly injective*, *exactly flat*) *semimodules*, which we call, for short, *e-projective* (resp., *e-injective*, *e-flat*) *semimodules* assuming that appropriate Hom and $\otimes$ functors preserve short exact sequences.

There are two main goals of this dissertation: The first goal is to investigate the notions of projective (resp., injective, flat) semimodules over a semiring and clarify the relation between them.The second goal is to use the notions of projective (resp. injective, flat) semimodules to provide homological characterizations of some special classes of semirings (e.g., k-Noetherian, k-Artinian, Semisimple and von Neumann regular semirings). It is worth mentioning that complete characterizations of semirings all of whose left *cyclic* semimodules are e-injective were obtained recently by Abuhlail et. al. [6].

This dissertation is divided into three chapters:

In Chapter One, we collect the basic definitions, examples and preliminaries used in this dissertation. Among others, we include the definitions and basic properties of *tensor products* and *exact sequences.* We also demonstrate that the tensor and the Hom functors build an adjoint pair. Moreover, we demonstrate the existence of *pullbacks* and *pushouts* (Theorem 1.47) in the category of semimodules over an arbitrary semiring. Although no explicit construction of the pushouts is given, we provide a description that is good enough to help us in proving several theorems in the sequel.

In Chapter Two, we investigate mainly the e-projective (resp., e-injective, e-flat) semimodules over a semiring and clarify their relations with the notions of projective (resp., injective, flat) semimodules as well as the so called k-projective (resp., i-injective, i-flat) semimodules.

Section 2.1 is devoted to projective semimodules. In Proposition 2.5, we demonstrate that every projective left semimodule is in fact e-projective. In Example 2.6, we show that the Boolean Algebra $\mathbb{B}$ considered as a $\mathbb{Q}^{+}$-semimodule in the canonical way is $\mathbb{Q}^{+}$-e-projective but not $\mathbb{Q}^{+}$-projective. A complete characterization of k-projective left semimodules through the right-splitting of short exact sequences is given in Proposition 2.9. In Lemma 2.11 and Proposition 2.12, we show that the class of e-projective left S-semimodules is closed under retracts and direct sums.

Section 2.2 is devoted to injective semimodules. Lemma 2.21 and Proposition 2.22 show that the class of injective left semimodules is closed under retracts and direct products. It was shown in [6, Proposition-Example 4.6.] that, for an additively idempotent division semiring D, the class of e-injective D-semimodules is *strictly* larger than the class of injective D-semimodules. Subsection 2.2.1 is devoted to showing that for the semiring $S := M_2(\mathbb{R}^{+})$, the class of S-i-injective left semimodules is strictly larger than the class of S-e-injective left S-semimodules. In particular, Lemma 2.26 shows that *all* left S-semimodules are i-injective, while Example 2.27 provides an example of a left S-semimodule which is not S-e-injective. While every module over a ring R can be embedded in an injective semimodules, and a module M is injective if M is R-injective (using the Baer's Criterion), any semiring whose category of semimodules has these nice properties is a *ring* [24, Theorem 3]. We investigate the so called *embedding problem*. Call a left S-semimodule *c-i-injective* if it is M-injective for every cancellative left S-semimodule M. We prove in Theorem 2.43 that every left S-semimodule can be embedded in a *c-i-injective* left S-semimodule.

Section 2.3 is devoted to flat semimodules. A flat semimodule is one which is the direct colimit of *finitely presented* semimodules [4]. It was proved by Abuhlail [4, Theorem 3.6] that flat left S-semimodules are e-flat. We prove in Lemma 2.51 and Proposition 2.52 that the class of e-flat left S-semimodules is closed under retracts and direct sums. In Theorem 2.57, we show that if S is a (left and right) subtractive semiring each of its right semimodules is S-e-flat, then S is a *von Neumann regular semiring.*

In Chapter Three, we use the notions of projective, injective, and flat semimodules to describe and characterize special classes of semirings.

Section 3.1 is devoted to left k-Noetherian (resp., left k-Artinian) semirings which satisfy ACC (resp., DCC) on left *subtractive* ideals, called also k-ideals. In Example 3.6, we show that $S := M_2(\mathbb{R}^+)$ is left k-Noetherian but not left Noetherian, and is left k-Artinian but not left Artinian. In Theorem 3.9, we show that if every subtractive left ideal of a semiring S is a direct summand, then S is left k-Artinian and left k-Noetherian. In Theorem 3.12, we show that if S is a semiring such that every left S-semimodule can be embedded into a left S-i-

injective semimodule (*e.g.,* S is an additively idempotent [6], or a cancellative semiring [18]) and if every direct sum of left S-i-injective left S-semimodules is S-i-injective, then S is left k-Noetherian.

Section 3.2 is devoted to ideal-semisimple and congruence-semisimple semirings. A semiring S is left (right) ideal-semisimple if S is a direct sum of ideal-simple (*i.e.* having $\{0\}$ as the only proper subsemimodule) left (right) ideals. By [20, Theorem 7.8], S is left ideal-semisimple (equivalently, right ideal-semisimple) if and only if $S \simeq M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$, where for each $i = 1, \cdots, k$, D_i is a division semiring, and $M_{n_i}(D_i)$ is the semiring of $n_i \times n_i$-matrices over D_i. A left subtractive semiring S was shown to be left ideal-semisimple if and only if all of its left semimodules are S-k-projective [30, Theorem 4.4]. In Theorem 3.29, we extend this (and other) characterization(s) to *commutative*, not necessarily subtractive, semirings satisfying some technical condition. In Theorem 3.36, we show that a *commutative* semiring S is ideal-semisimple if and only if every S-semimodule is S-e-injective (S-k-injective) and S satisfies some technical condition. The two results are combined in Theorem 3.37 to provide a complete characterization of commutative ideal-semisimple semirings. The congruence-semisimple version of this main result is given in Theorem 3.38. Examples 3.33 and 3.39 demonstrate that the conditions assumed in our main results in this section, in particular the commutativity of the base semiring, cannot be dropped.

CHAPTER 1

SEMIRINGS AND SEMIMODULES

1.1 Preliminaries

In this section, we provide the basic definitions and preliminaries used in this work. Any notions that are not defined can be found in our main reference [18].

1.1.1 Basic Definitions and Examples

Definition 1.1 *([18]) A* ***semiring*** *is a datum* $(S, +, 0, \cdot, 1)$ *consisting of a non-empty set* S *along with two binary operations "+" (addition) and "·" (multiplication) such that:*

(1) $(S, +, 0)$ *is a commutative monoid with neutral element* 0*;*

(2) $(S, \cdot, 1)$ *is a monoid with neutral element* 1*;*

(3) $0 \neq 1$;

(4) $a \cdot 0 = 0 = 0 \cdot a$ *for all* $a \in S$;

(5) *For all* $a, b, c \in S$ *we have*

$$a(b+c) = ab + ac \text{ and } (a+b)c = ac + bc.$$

Definitions 1.1 *([18]) Let* $(S, +, 0, \cdot, 1)$ *be a semiring.*

- *If* $\{0,1\} \subseteq S' \subseteq S$ *and* S' *is closed under the two binary operations "+" and "·", then we say that* S' *is a subsemiring of* S.
- *If the monoid* $(S, \cdot, 1)$ *is commutative, we say that* S *is a* commutative semiring.
- *If* $(S \backslash \{0\}, \cdot, 1)$ *is a group, we say* S *is a* division semiring.
- *A commutative division semiring is called a* semifield.
- *The set of* additively idempotent elements of S *is defined as*

$$I^{+}(S) := \{s \in S \mid \ s + s = s\}. \tag{1.1}$$

  If $I^{+}(S) = S$, *we say that* S *is* additively idempotent.
- *The set of* multiplicatively idempotent elements of S *is defined as*

$$I^{\times}(S) := \{s \in S \mid \ s \cdot s = s\}. \tag{1.2}$$

If $I^{\times}(S) = S$, *we say that* S *is* multiplicatively idempotent.

- *The set of* idempotent elements of S *is defined as*

$$I(S) := I^{+}(S) \cap I^{\times}(S). \tag{1.3}$$

We say that S *is an idempotent semiring if* $I(S) = S$.

- *Let*

$$V(S) := \{s \in S \mid s + t = 0 \text{ for some } t \in S\}. \tag{1.4}$$

If $V(S) = \{0\}$, *we say that* S *is* zerosumfree. *Notice that* $V(S) = S$ *if and only if* S *is a ring.*

- *The set of* cancellative elements of S *is defined as*

$$K^{+}(S) = \{x \in S \mid x + y = x + z \Longrightarrow y = z \text{ for any } y, z \in S\}.$$

We say that S *is a* cancellative semiring *if* $K^{+}(S) = S$.

- *We say that* $s \in S$ *is a left (right)* zero divisor *if* $st = 0$ $(ts = 0)$ *for some* $t \in S \backslash \{0\}$. *If* S *has no non-zero zero-divisors, we say that* S *is* entire.

- *If* $a \in S$ *is such that* $s + a = a$ *for all* $s \in S$, *then we say that* a *is an* infinite element *of* S. *If* S *has an infinite element, then it is* unique.

- *If* $a \in S$ *is an infinite element such that* $sa = a = as$ *for all* $s \in S \backslash \{0\}$, *then we say that* a *is a* strongly infinite element.

- *The* zeroid *of S is defined as*

$$Z(S) = \{z \in S \mid \ z + s = z \textit{ for some } s \in S\}. \tag{1.5}$$

We say that S is a zeroic semiring *if $Z(S) = S$; otherwise S is* nonzeroic. *On the other hand, we say that S is a* plain semiring *if $Z(S) = \{0\}$; otherwise S is* nonplain.

Examples 1.2 *([18])*

- *Every ring is a cancellative semiring.*

- *Any* distributive bounded lattice $\mathcal{L} = (L, \vee, 1, \wedge, 0)$ *is a commutative idempotent semiring and 1 is an infinite element of $\mathcal{L}$.*

- *Let R be any ring. The set $\mathcal{I} = (Ideal(R), +, 0\cdot, R)$ of ideals of R is a zerosumfree semiring and R is a strongly infinite element of $\mathcal{I}$.*

- *The set $(\mathbb{Z}^+, +, 0, \cdot, 1)$ of non-negative integers is a commutative cancellative zerosumfree entire semiring which is not a ring.*

- *The set $(\mathbb{R}^+, +, 0, \cdot, 1)$ of non-negative real numbers is a semifield. The subset $(\mathbb{Q}^+, +, 0, \cdot, 1)$ of non-negative rational numbers is a subsemifield of $\mathbb{R}^+$, and $\mathbb{Z}^+$ is subsemiring of $\mathbb{Q}^+$.*

- *$M_n(S)$, the set of all $n \times n$ matrices over a (zerosumfree) semiring S, is a (zerosumfree) semiring.*

- *The Boolean algebra* $\mathbb{B} := \{0, 1\}$ *with* $1 + 1 = 1$ *is an idempotent semifield which is not a field and* 1 *is a strongly infinite element of* $\mathbb{B}$.

- *The* max-plus algebra $\mathbb{R}_{\max,+} := (\mathbb{R} \cup \{-\infty\}, \max, -\infty, +, 0)$ *is an additively idempotent semiring.*

- *The* min-plus algebra $\mathbb{R}_{\min,+} := (\mathbb{R} \cup \{\infty\}, \min, \infty, +, 0)$ *is a additively idempotent semiring.*

- *The* max-min algebra $\mathbb{R}_{\max,\min} := (\mathbb{R} \cup \{-\infty, \infty\}, \max, -\infty, \min, \infty)$ *is an idempotent semiring and* ∞ *is the infinite element of* $\mathbb{R}_{\max,\min}$.

- *The* log algebra $(\mathbb{R} \cup \{-\infty, \infty\}, \oplus, \infty, +, 0)$ *is a semiring, where*

$$x \oplus y = -ln(e^{-x} + e^{-y})$$

Example 1.3 *([18, Example 1.4]) Let* R *be a commutative integral domain. Notice that* $\mathcal{I} = (Ideal(D), +, 0, \cap, D)$ *of ideals of* D *is a bounded lattice. Moreover,* $\mathcal{I}$ *is a distributive lattice if and only if* D *is a Prüfer domain (*i.e. *if every finitely generated ideal is projective). If this case,* $\mathcal{I}$ *is a zerosumfree idempotent semiring, and the subset of finitely generated ideals forms a subsemiring.*

Example 1.4 *([18, Example 1.8], [8]) Consider*

$$B(n, i) := (B(n, i), \oplus, 0, \odot, 1),$$

where $B(n, i) = \{0, 1, 2, ..., n - 1\}$ *and*

$a \oplus b = a + b$ if $a + b < n$; otherwise, $a \oplus b = c$ with $i \leq c < n$ is the unique natural number satisfying $c \equiv a + b \bmod (n - i)$;

$a \odot b = ab$ if $ab < n$; otherwise, $a \odot b = c$ with $i \leq c < n$ is the unique natural number with $c \equiv ab \bmod (n - i)$.

Then $B(n, i)$ is a semiring. Notice that $B(n, 0) = \mathbb{Z}_n$ (a group) and that $B(2, 1) = \mathbb{B}$ (the Boolean Algebra).

Definition 1.2 *[18, page 149, 156] Let S be a semiring. A **left** S-**semimodule** is a commutative monoid $(M, +, 0_M)$ with a map (called* scalar multiplication*)*

$$S \times M \to M, \ (s, m) \mapsto sm,$$

which satisfies the following conditions for all $m, m_1, m_2 \in M$ and $s, s_1, s_2 \in S$:

(1) $(s_1 s_2)m = s_1(s_2 m)$;

(2) $(s_1 + s_2)m = s_1 m + s_2 m$,

(3) $s(m_1 + m_2) = sm_1 + sm_2$;

(4) $1_S m = m$;

(5) $s0_M = 0_M = 0_S m$.

*If M is a left S-semimodule, and $(L, +, 0_M) \leq (M, +, 0)$, is a submonoid such that $sl \in L$ for all $s \in S$ and $l \in L$, then we say that L is an S-**subsemimodule** of M and write $L \leq_S M$.*

*For two left S-semimodules M and N, a map $f : M \longrightarrow N$ is an S-**linear map** if it preserves the addition and the scalar multiplication. The set $Hom_S(M, N)$ of all S-linear maps from M to N is a commutative monoid under the usual addition of maps. The category of left S-semimodules and S-linear maps is denoted by ${}_S\mathbf{SM}$. The category $\mathbf{SM}_S$ of right S-semimodules is defined analogously.*

Definition 1.3 *Let S and T be semirings. If M is a left S-semimodule and a right T-semimodule such that $(sm)t = s(mt)$ for all $s \in S$, $t \in T$ and $m \in M$, we say that M is an (S, T)**-bisemimodule**. The category of (S, T)-bisemimodules, with arrows being the left S-linear right T-linear maps, is denoted by ${}_S\mathbf{SM}_T$.*

Definition 1.4 *Let S be a semiring. A left (resp., right) ideal of S can be defined as an S-subsemimodule of ${}_SS$ (resp., of S_S). A (two-sided) ideal of S can be defined as an (S, S)-subbisemimodule of ${}_SS_S$.*

Definition 1.5 *Let S be a semiring and M a left S-semimodule. The subsets $I^+(M)$ (resp., $V(M)$, $K^+(M)$, $Z(M)$) of M are defined in a way analogous to that defined for the semiring S, and we call M an **additively idempotent semimodule** (resp., **zerosumfree semimodule**, **cancellative semimodule**, **zeroic semimodule, plain semimodule**) if $I^+(M) = M$ (resp., $V(M) = 0$, $K^+(M) = M$, $Z(M) = M$, $Z(M) = \{0_M\}$).*

Example 1.5 *The category of $\mathbb{Z}^+$-semimodules is nothing but the category of commutative monoids.*

Example 1.6 *Let* $(S, +, 0, \cdot, 1)$ *be a semiring. Then* S *and* $S^{(\Lambda)}$ *(the direct sum of* S *over a non-empty index set* Λ*) are* (S, S)*-bisemimodules with left and right actions induced by "*$\cdot$*".*

Examples 1.7 *Consider the semiring* $M_2(\mathbb{R}^+)$*-semimodules. Then*

$$E_1 = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\} \text{ and } E_2 = \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\}$$

are left $M_2(\mathbb{R}^+)$*-semimodules.*

Example 1.8 *([18, page 150, 154]) Let* S *be a semiring,* M *be a left* S*-semimodule and* $L \subseteq M$*. The* ***subtractive closure*** *of* L *is defined as*

$$\overline{L} := \{m \in M \mid \ m + l = l' \text{ for some } l, l' \in L\}. \tag{1.6}$$

We say that L *is subtractive if* $L = \overline{L}$*. The left* S*-semimodule* M *is a* ***subtractive semimodule*** *if every* S*-subsemimodule* $L \leq_S M$ *is subtractive.*

Definition 1.6 *[18, page 71] Let* S *be a semiring. We say that* S *is a* ***left subtractive semiring*** *(****right subtractive semiring****) if every left (right) ideal of* S *is subtractive. We say that* S *is a subtractive semiring if* S *is both left and right subtractive.*

Remark 1.9 *Whether a left subtractive semiring is necessarily right subtractive was an open problem till a counterexample was given in [29, Fact 2.1].*

Definition 1.7 *[18, page 162] Let S be a semiring. An equivalence relation ρ on a left S-semimodule M is a* ***congruence relation*** *if it preserves the addition and the scalar multiplication on M,* i.e. *for all $s \in S$ and $m, m', n, n' \in M$:*

$$m\rho m' \text{ and } n\rho n' \Longrightarrow (m+m')\rho(n+n'),$$

$$m\rho m' \Longrightarrow (sm)\rho(sm').$$

Example 1.10 *Let S be a semiring, M a left S-semimodule and $N \leq_S M$. The* ***Bourne relation*** *$\equiv_N$ on M is defined as:*

$$m \equiv_N m' \Leftrightarrow m+n = m'+n' \text{ for some } n, n' \in N.$$

It is clear that $\equiv_N$ is a congruence relation. Moreover, $M/N = M/\equiv_N = \{[m]_N \mid m \in M\}$ $(= M/\overline{N})$ is a left S-semimodule, the canonical surjective map $\pi_N : M \longrightarrow M/N$ is S-linear, and $Ker(\pi_N) = \overline{N}$. In particular, $Ker(\pi_N) = 0$ if and only if $N \leq_S M$ is subtractive (this explains why subtractive ideals are called k-ideals in many references).

Following [16], we use the following definitions.

Definition 1.8 *Let S be a semiring. A left S-semimodule M is*

ideal-simple *if 0 and M are the only S-subsemimodules of M;*

congruence-simple *if* $M \times M$ *and*

$$\Delta_M := \{(m, m) \mid m \in M\}$$

are the only congruence relations on M.

Remark 1.11 *If* M *is a congruence-simple left* S*-semimodule, then the only subtractive* S*-subsemimodules of* M *are* 0 *and* M. *To show this, suppose that* $N \neq 0$ *is a subtractive* S*-subsemimodule of* M. *Then* $\equiv_N$ *is a congruence relation on* M *with* $n \equiv_N 0$ *for some* $n \in N \backslash 0$. *Thus* $\equiv_N \neq \Delta_M$, *which implies* $\equiv_N = M^2$ *as* M *is congruence-simple. If* $m \in M$, *then* $mM^2 0$, *that is* $m \equiv_N 0$. *Therefore, there exist* $n, n' \in N$ *such that* $m + n = n'$. *Since* N *subtractive,* $m \in N$. *Hence* $M = N$.

Example 1.12 *[31, 3.7 (b)] Let* $(M, +, 0)$ *be a finite lattice that is not distributive. The endomorphism semiring* $\mathbf{E}_M$ *of* M *is a congruence-simple semiring which is* not *ideal-simple.*

Example 1.13 *([31, 3.7 (c)]) Every zerosumfree division semiring that is not isomorphic to* $\mathbb{B}$ *(e.g.,* $\mathbb{R}^+$*) is left ideal-simple but* not *left congruence-simple. Notice that* D *is ideal-simple as the only left ideals of* D *are* $\{0\}$ *and* D. *On the other hand, if* D *is not isomorphic to* $\mathbb{B}$, *then*

$$\rho = \{(a, b) |\ a, b \in D \backslash \{0\}\} \cup \{(0, 0)\}$$

is a non-trivial non-universal congruence relation on ${}_D D$.

Lemma 1.14 *A left S-semimodule M is congruence-simple if and only if every non-zero S-linear map from M is injective.*

Proof. ($\Rightarrow$) Let $f : M \to N$ be a non-zero S-linear map and pick some $m \in M \backslash \{0\}$ such that $f(m) \neq 0$. Since $\equiv_f$ is a congruence relation on M with $m \not\equiv_f 0$, we know $\equiv_f \neq M^2$. It follows that $\equiv_f = \Delta_M$ as M is congruence-simple. Hence f is injective.

($\Leftarrow$) Assume that M is congruence-simple. Let ρ be a congruence relation on M. The canonical map $f : M \to M/\rho$ is S-linear. If $f = 0$, then $[m]_\rho = [0]_\rho$ for every $m \in M$, that is $m\rho 0$ for every $m \in M$ and $m\rho m'$ for every $m, m' \in M$. If $f \neq 0$, then f is injective, that is $[m]_\rho \neq [m']_\rho$ whenever $m \neq m'$. Thus $m \not\rho\, m'$ whenever $m \neq m'$ and $\rho = \Delta_M$.■

Lemma 1.15 *A left S-semimodule M is ideal-simple if and only if every non-zero S-linear map to M is surjective.*

Proof. ($\Rightarrow$) Let $f : L \to M$ be a non-zero S-linear map. Then there exists $l \in L \backslash 0$ such that $f(l) \neq 0$. Thus, $f(L)$ is a non-zero subsemimodule of M and so $f(L) = M$ as M ideal-simple.

($\Leftarrow$) Let K be a subsemimodule of M. Then the embedding $f : K \to M$ is an S-linear map. If $f = 0$, then $K = f(K) = 0$. If $f \neq 0$, then f is surjective, that is $K = f(K) = M$.■

1.16 *([7]) The category ${}_S\mathbf{SM}$ of left semimodules over a semiring S is a* variety *in the sense of Universal Algebra (closed under homomorphic images, subobjects* *and arbitrary products). Whence ${}_S\mathbf{SM}$ is complete, i.e. has all limits (e.g., direct products, equalizers, kernels, pullbacks, inverse limits) and cocomplete, i.e. has all colimits (e.g., direct coproducts, coequalizers, cokernels, pushouts, direct colimits).*

Definition 1.9 *([18, page 184]) Let S be a semiring. A left S-semimodule M is the **direct sum** of a family $\{L_\lambda\}_{\lambda\in\Lambda}$ of S-subsemimodules $L_\lambda \leq_S M$, and we write $M = \bigoplus\limits_{\lambda\in\Lambda} L_\lambda$, if every $m \in M$ can be written in a* unique way *as a finite sum $m = l_{\lambda_1} + \cdots + l_{\lambda_k}$ where $l_{\lambda_i} \in L_{\lambda_i}$ for each $i = 1, \cdots, k$. Equivalently, $M = \bigoplus\limits_{\lambda\in\Lambda} L_\lambda$ if $M = \sum\limits_{\lambda\in\Lambda} L_\lambda$ and for each finite subset $A \subseteq \Lambda$ with $l_a, l'_a \in L_a$, we have:*

$$\sum_{a\in A} l_a = \sum_{a\in A} l'_a \Longrightarrow l_a = l'_a \text{ for all } a \in A.$$

1.17 *An S-semimodule N is a **retract** of an S-semimodule M if there exists a (surjective) S-linear map $\theta : M \longrightarrow N$ and an (injective) S-linear map $\psi : N \longrightarrow M$ such that $\theta \circ \psi = \mathrm{id}_N$ (equivalently, $N \simeq \alpha(M)$ for some* idempotent *endomorphism $\alpha \in \mathrm{End}(M_S)$).*

1.18 *An S-semimodule N is a **direct summand** of an S-semimodule M (*i.e. *$M = N \oplus N'$ for some S-subsemimodule N' of M) if and only if there exists $\alpha \in \mathrm{Comp}(\mathrm{End}(M_S))$ s.t. $\alpha(M) = N$ where for any semiring T we set*

$$\mathrm{Comp}(T) = \{t \in T \mid \ \exists\, \widetilde{t} \in T \text{ with } t + \widetilde{t} = 1_T \text{ and } t\widetilde{t} = 0_T = \widetilde{t}t\}.$$

Indeed, every direct summand of M is a retract of M; the converse is not true in general; for example N_1 in Example 2.27 is a retract of $M_2(\mathbb{R}^+)$ that is not a *direct summand. Golan [18, Proposition 16.6] provided characterizations of direct summands.*

Remarks 1.19 *Let M be a left S-semimodule and $K, L \leq_S M$ be S-semimodules of M.*

(1) *If $K + L$ is direct, then $K \cap L = 0$. The converse is not true in general.*

(2) *If $M = K \oplus L$, then $M/K \simeq L$.*

Example 1.20 *Let $S = M_2(\mathbb{R}^+)$. Notice that*

$$E_1 = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\} \text{ and } N_{\geq 1} = \left\{ \begin{bmatrix} a & c \\ b & b \end{bmatrix} \mid a \leq c, b \leq d, a, b, c, d \in \mathbb{R}^+ \right\}$$

are left ideals of S with $E_1 \cap N_{\geq 1} = \{0\}$. However, the sum $E_1 + N_{\geq 1}$ is not direct since

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Definition 1.10 *Let S be a semiring. A left S-semimodule M is*

ideal-semisimple *if $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$, a direct sum of ideal-simple S-subsemimodules;*

congruence-semisimple *if $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$, a direct sum of congruence-simple S-subsemimodules.*

1.21 *([28, 3.1], [4, 2.1]) Let L be a right S-semimodule and M be a left S-semimodule and let A be a commutative monoid. We call $f : L \times M \to A$ an*

*S-**balanced** map if for all $l, l' \in L$, $m, m' \in M$ and $s \in S$ we have*

$$f(l,m)+f(l,m') = f(l,m+m'), f(l,m)+f(l',m) = f(l+l',m),\ \ f(ls,m) = f(l,sm).$$

*A **tensor product** of M and L is a commutative monoid $L \otimes_S M$ along with an S-balanced map $h : L \times M \to L \otimes_S M$ satisfying the* Universal Property of Tensor Products*: whenever $f : L \times M \to A$ is an S-balanced map, there exists a unique S-balanced map $\varphi : L \otimes_S M \to A$ such that $\varphi \circ h = f$.*

$$\begin{tikzcd}
L \times M \arrow[rr, "h"] \arrow[dr, "f"'] & & L \otimes_S M \arrow[dl, dashed, "\varphi"] \\
& A &
\end{tikzcd}$$

Technically, $L \otimes_S M = F/\rho$, where F is the free commutative monoid with basis $L \times M$ (every element of F can be written uniquely as a linear combination of elements of the set $\{\delta_{(l,m)} |\ (l,m) \in L \times M\}$ where $\delta_{(l,m)}$ is the Kronecker delta function) and ρ is the congruence relation on F generated by

$$\{(\delta_{(l+l',m)}, \delta_{(l,m)} + \delta_{(l',m)}), (\delta_{(l,m+m')}, \delta_{(l,m)} + \delta_{(l,m')}), (\delta_{(ls,m)}, \delta_{(l,sm)})\}.$$

We will write

$$[\sum_j \delta_{(l_j,m_j)}]_\rho = \sum_j (l_j \otimes_S m_j).$$

Lemma 1.22 *For every right S-semimodule M, there exists a natural right S-*

isomorphism

$$\theta_M : M \otimes_S S \to M,\ \ m \otimes_S s \mapsto ms.$$

1.1.2 Exact Sequences

Throughout, $(S, +, 0, \cdot, 1)$ is a semiring and, unless otherwise explicitly mentioned, an S-module is a **left** S-semimodule.

Definition 1.11 *A morphism of left S-semimodules $f : L \to M$ is*

*k-**normal** if whenever $f(m) = f(m')$ for some $m, m' \in M$, we have $m + k = m' + k'$ for some $k, k' \in Ker(f)$;*

*i-**normal** if* $\mathrm{Im}(f) = \overline{f(L)}$ *($:= \{m \in M |\ m + l \in L$ for some $l \in L\}$).*

***normal** if f is both k-normal and i-normal.*

Remarks 1.23 (1) *Among others, Takahashi ([37]) and Golan [18] called k-normal (resp., i-normal, normal) S-linear maps k-*regular *(resp., i-*regular*,* regular*) morphisms. We changed the terminology to avoid confusion with the regular monomorphisms and regular epimorphisms in Category Theory which have different meanings when applied to categories of semimodules.*

(2) *Our terminology is consistent with Category Theory noting that: every surjective S-linear map is i-normal, whence the k-normal surjective S-linear map are normal and are precisely the so-called **normal epimorphisms**. On the other hand, the injective S-linear maps are k-normal, whence the*

i-normal injective S-linear maps are normal and are precisely the so called ***normal monomorphisms*** *(see [1]).*

Lemma 1.24 *Let $L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence of semimodules.*

(1) *Let g be injective.*

   (a) f is k-normal if and only if $g \circ f$ is k-normal.

   (b) If $g \circ f$ is i-normal (normal), then f is i-normal (normal).

   (c) Assume that g is i-normal. Then f is i-normal (normal) if and only if $g \circ f$ is i-normal (normal).

(2) *Let f be surjective.*

   (a) g is i-normal if and only if $g \circ f$ is i-normal.

   (b) If $g \circ f$ is k-normal (normal), then g is k-normal (normal).

   (c) Assume that f is k-normal. Then g is k-normal (normal) if and only if $g \circ f$ is k-normal (normal).

Proof.

(1) Let g be injective; in particular, g is k-normal.

   (a) Assume that f is k-normal. Suppose that $(g \circ f)(l_1) = (g \circ f)(l_2)$ for some $l_1, l_2 \in L$. Since g is injective, $f(l_1) = f(l_2)$. By assumption, there exist $k_1, k_2 \in \mathrm{Ker}(f)$ such that $l_1 + k_1 = l_2 + k_2$. Since $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(g \circ f)$, we conclude that $g \circ f$ is k-normal. On the other hand, assume that

$g \circ f$ is k-normal. Suppose that $f(l_1) = f(l_2)$ for some $l_1, l_2 \in L$. Then $(g \circ f)(l_1) = (g \circ f)(l_2)$ and so there exist $k_1, k_2 \in \mathrm{Ker}(g \circ f)$ such that $l_1 + k_1 = l_2 + k_2$. Since g is injective, $\mathrm{Ker}(g \circ f) = \mathrm{Ker}(f)$ whence f is k-normal.

(b) Assume that $g \circ f$ is i-normal. Let $m \in \overline{f(L)}$, so that $m + f(l_1) = f(l_2)$ for some $l_1, l_2 \in L$. Then $g(m) \in \overline{(g \circ f)(L)} = (g \circ f)(L)$. Since g is injective, $m \in f(L)$. So, f is i-normal.

(c) Assume that g and f are i-normal. Let $n \in \overline{(g \circ f)(L)}$, so that $n + g(f(l_1)) = g(f(l_2))$ for some $l_1, l_2 \in L$. Since g is i-normal, $n \in g(M)$ say $n = g(m)$ for some $m \in M$. But g is injective, whence $m + f(l_1) = f(l_2)$, i.e. $m \in \overline{f(L)} = f(L)$ since f is i-normal by assumption. So, $n = g(m) \in (g \circ f)(L)$. We conclude that $g \circ f$ is i-normal.

(2) Let f be surjective; in particular, f is i-normal.

(a) Assume that g is i-normal. Let $n \in \overline{(g \circ f)(L)}$ so that $n + g(f(l_1)) = g(f(l_2))$ for some $l_1, l_2 \in L$. Since g is i-normal, $n = g(m)$ for some $m \in M$. Since f is surjective, $n = g(m) \in (g \circ f)(L)$. So, $g \circ f$ is i-normal.

On the other hand, assume that $g \circ f$ is i-normal. Let $n \in \overline{g(M)}$, so that $n + g(m_1) = g(m_2)$ for some $m_1, m_2 \in M$. Sine f is surjective, there exist $l_1, l_2 \in L$ such that $f(l_1) = m_1$ and $f(l_2) = m_2$. Then, $n + (g \circ f)(l_1) = (g \circ f)(l_2)$, i.e. $n \in \overline{(g \circ f)(L)} = (g \circ f)(L) \subseteq g(M)$. So, g is i-normal.

(b) Assume that $g \circ f$ is k-normal. Suppose that $g(m_1) = g(m_2)$ for some $m_1, m_2 \in M$. Since f is surjective, we have $(g \circ f)(l_1) = (g \circ f)(l_2)$ for some $l_1, l_2 \in L$. By assumption, $g \circ f$ is k-normal and so there exist $k_1, k_2 \in \mathrm{Ker}(g \circ f)$ such that $l_1 + k_1 = l_2 + k_2$ whence $m_1 + f(k_1) = m_2 + f(k_2)$. Indeed, $f(k_1), f(k_2) \in \mathrm{Ker}(g)$. i.e. g is k-normal.

(c) Assume that f and g are k-normal. Suppose that $(g \circ f)(l_1) = (g \circ f)(l_2)$ for some $l_1, l_2 \in L$. Since g is k-normal, we have $f(l_1) + k_1 = f(l_2) + k_2$ for some $k_1, k_2 \in \mathrm{Ker}(g)$. But f is surjective; whence $k_1 = f(l_1')$ and $k_2 = f(l_2')$ for some $l_1', l_2' \in L$, i.e. $f(l_1 + l_1') = f(l_2 + l_2')$. Since f is k-normal, $l_1 + l_1' + k_1' = l_2 + l_2' + k_2'$ for some $k_1', k_2' \in \mathrm{Ker}(f)$. Indeed, $l_1' + k_1'$, $l_2' + k_2' \in \mathrm{Ker}(g \circ f)$. We conclude that $g \circ f$ is k-normal.■

The proof of the following lemma is straightforward:

Lemma 1.25 (1) *Let* $\{f_\lambda : L_\lambda \longrightarrow M_\lambda\}_\Lambda$ *be a family of left* S*-semimodule morphisms and consider the induced* S*-linear map* $f : \bigoplus_{\lambda \in \Lambda} L_\lambda \longrightarrow \bigoplus_{\lambda \in \Lambda} M_\lambda$. *Then* f *is normal (resp. k-normal, i-normal) if and only if* f_λ *is normal (resp. k-normal, i-normal) for every* $\lambda \in \Lambda$.

(2) *A morphism* $\varphi : L \longrightarrow M$ *of left* S*-semimodules is normal (resp. k-normal, i-normal) if and only if* $\mathrm{id}_F \otimes_S \varphi : F \otimes_S L \longrightarrow F \otimes_S M$ *is normal (resp. k-normal, i-normal) for every non-zero free right* S*-semimodule* F.

(3) *If* P_S *is projective and* $\varphi : L \longrightarrow M$ *is a normal (resp. k-normal, i-normal) morphism of left* S*-semimodules, then* $\mathrm{id}_F \otimes_S \varphi : P \otimes_S L \longrightarrow P \otimes_S M$ *is*

normal (resp. k-normal, i-normal).

There are several notions of exactness for sequences of semimodules. In this thesis, we use the relatively new notion introduced by Abuhlail:

Definition 1.12 *([1, 2.4]) A sequence*

$$L \xrightarrow{f} M \xrightarrow{g} N \tag{1.7}$$

of left S-semimodules is ***exact*** *if g is k-normal and $f(L) = Ker(g)$.*

1.26 *We call a sequence of S-semimodules*

$$L \overset{f}{\to} M \overset{g}{\to} N$$

proper-exact *if $f(L) = \mathrm{Ker}(g)$;*

semi-exact *if $\overline{f(L)} = \mathrm{Ker}(g)$;*

quasi-exact *if $\overline{f(L)} = \mathrm{Ker}(g)$ and g is k-normal.*

1.27 *We call a (possibly infinite) sequence of S-semimodules*

$$\cdots \to M_{i-1} \overset{f_{i-1}}{\to} M_i \overset{f_i}{\to} M_{i+1} \overset{f_{i+1}}{\to} M_{i+2} \to \cdots \tag{1.8}$$

chain complex *if $f_{j+1} \circ f_j = 0$ for every j;*

exact *(resp.,* proper-exact, semi-exact, quasi-exact*) if each partial sequence with three terms $M_j \overset{f_j}{\to} M_{j+1} \overset{f_{j+1}}{\to} M_{j+2}$ is exact (resp., proper-exact, semi-exact, quasi-exact).*

*A **short exact sequence** (or a **Takahashi extension** [35]) of S-semimodules is an exact sequence of the form*

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

Remark 1.28 *In the sequence (1.7), the inclusion $f(L) \subseteq Ker(g)$ forces $f(L) \subseteq \overline{f(L)} \subseteq Ker(g)$, whence the assumption $f(L) = Ker(g)$ guarantees that $f(L) = \overline{f(L)}$,* i.e. *f is i-normal. So, the definition puts conditions on f and g that are dual to each other (in some sense).*

The follows examples show some of the advantages of the new definition of exact sequences over the old ones:

Lemma 1.29 *Let L, M and N be S-semimodules.*

(1) $0 \longrightarrow L \xrightarrow{f} M$ *is exact if and only if f is injective.*

(2) $M \xrightarrow{g} N \longrightarrow 0$ *is exact if and only if g is surjective.*

(3) $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$ *is semi-exact and f is normal if and only if* $L \simeq \mathrm{Ker}(g)$.

(4) $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$ *is exact if and only if* $L \simeq \mathrm{Ker}(g)$ *and g is k-normal.*

(5) $L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ *is semi-exact and g is normal if and only if $N \simeq M/f(L)$.*

(6) $L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ *is exact if and only if $N \simeq M/f(L)$ and f is i-normal.*

(7) $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ *is exact if and only if* $L \simeq \mathrm{Ker}(g)$ *and* $N \simeq M/L$.

Corollary 1.30 *The following assertions are equivalent:*

(1) $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *is an exact sequence of* S*-semimodules;*

(2) $L \simeq \mathrm{Ker}(g)$ *and* $N \simeq M/f(L)$*;*

(3) f *is injective,* $f(L) \simeq \mathrm{Ker}(g)$*,* g *is surjective and (k-)normal.*

In this case, f *and* g *are normal morphisms.*

Remark 1.31 *A morphism of semimodules* $\gamma : X \longrightarrow Y$ *is an isomorphism if and only if* $0 \longrightarrow X \xrightarrow{\gamma} Y \longrightarrow 0$ *is exact if and only if* γ *is a normal bimorphism (*i.e. γ *is a normal monomorphism and a normal epimorphism). The assumption on* γ *to be normal cannot be removed here. For example, the embedding* $\iota : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$ *is a bimorphism of commutative monoids (*$\mathbb{Z}^+$*-semimodules) which is not an isomorphism. Notice that* ι *is not i-normal; in fact* $\overline{\iota(\mathbb{Z}^+)} = \mathbb{Z}$.

Remark 1.32 *An* S*-linear map is a monomorphism if and only if it is injective. Every surjective* S*-linear map is an epimorphism. The converse is not true in general.*

Example 1.33 *The embedding* $\iota : \mathbb{Z}^+ \to \mathbb{Z}$ *is a monoid epimorphism as* $(f \circ \iota)(1_{\mathbb{Z}^+}) = (g \circ \iota)(1_{\mathbb{Z}^+})$ *implies* $f(1_{\mathbb{Z}}) = g(1_{\mathbb{Z}})$ *and* $f = g$ *for every monoid morphisms* $f, g : \mathbb{Z} \to M$*. However, it is clear that* ι *is not surjective.*

Lemma 1.34 (Compare with [37, Proposition 4.3.]) *Let $\gamma : X \to Y$ be a morphism of S-semimodules.*

(1) *The sequence*

$$0 \to \mathrm{Ker}(\gamma) \overset{\ker(\gamma)}{\longrightarrow} X \overset{\gamma}{\to} Y \overset{\mathrm{coker}(\gamma)}{\longrightarrow} \mathrm{Coker}(\gamma) \to 0 \tag{1.9}$$

with canonical S-linear maps is semi-exact. Moreover, (1.9) is exact if and only if γ is normal.

(2) *We have two exact sequences*

$$0 \to \overline{\gamma(X)} \overset{\ker(\mathrm{coker}(\gamma))}{\longrightarrow} Y \overset{\mathrm{coker}(\gamma)}{\longrightarrow} Y/\gamma(X) \to 0.$$

and

$$0 \to \mathrm{Ker}(\gamma) \overset{\ker(\gamma)}{\longrightarrow} X \overset{\mathrm{coker}(\ker(\gamma))}{\longrightarrow} X/\mathrm{Ker}(\gamma) \to 0.$$

Corollary 1.35 (Compare with [37, Proposition 4.8.]) *Let M be an S-semimodule.*

(1) *Let ρ an S-congruence relation on M and consider the sequence of S-semimodules*

$$0 \longrightarrow \mathrm{Ker}(\pi_\rho) \overset{\iota_\rho}{\longrightarrow} M \overset{\rho}{\longrightarrow} M/\rho \longrightarrow 0.$$

(a) $0 \to \mathrm{Ker}(\pi_\rho) \overset{\iota_\rho}{\longrightarrow} M \overset{\pi_\rho}{\longrightarrow} M/\rho \to 0$ *is exact.*

(b) $M/\rho = \mathrm{Coker}(\iota_\rho)$.

(2) *Let L be an S-subsemimodule of M.*

(a) *The sequence* $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi_L} M/L \to 0$ *is semi-exact.*

(b) $0 \to \overline{L} \xrightarrow{\iota} M \xrightarrow{\pi_L} M/L \to 0$ *is exact.*

(c) *The following assertions are equivalent:*

i. $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi_L} M/L \to 0$ *is exact;*

ii. $L = \mathrm{Ker}(\pi_L)$;

iii. $0 \longrightarrow L \xrightarrow{\iota} \overline{L} \longrightarrow 0$ *is exact;*

iv. *L is a subtractive subsemimodule.*

1.2 Adjoint pairs of Functors

Proposition 1.36 (*cf.* [14, Proposition 3.2.2]) *Let $\mathfrak{C}, \mathfrak{D}$ be arbitrary categories and $\mathfrak{C} \xrightarrow{F} \mathfrak{D} \xrightarrow{G} \mathfrak{C}$ be functors such that (F, G) is an adjoint pair.*

(1) *F preserves all colimits which turn out to exist in $\mathfrak{C}$.*

(2) *G preserves all limits which turn out to exist in $\mathfrak{D}$.*

Corollary 1.37 *Let S, T be semirings and ${}_TF_S$ a (T, S)-bisemimodule.*

(1) $F \otimes_S - : {}_S\mathbf{SM} \longrightarrow {}_T\mathbf{SM}$ *preserves all colimits.*

(a) *For every family of left S-semimodules $\{X_\lambda\}_\Lambda$, we have a canonical isomorphism of left T-semimodules*

$$F \otimes_S \bigoplus_{\lambda \in \Lambda} X_\lambda \simeq \bigoplus_{\lambda \in \Lambda} (F \otimes_S X_\lambda).$$

(b) *For any directed system of left S-semimodules $(X_j, \{f_{jj'}\})_J$, we have an isomorphism of left T-semimodules*

$$F \otimes_S \varinjlim X_j \simeq \varinjlim (F \otimes_S X_j).$$

(c) $F \otimes_S -$ *preserves coequalizers.*

(d) $F \otimes_S -$ *preserves cokernels.*

(2) $\mathrm{Hom}_T(F, -) : {}_T\mathbf{SM} \longrightarrow {}_S\mathbf{SM}$ *preserves all limits.*

(a) *For every family of left T-semimodules $\{Y_\lambda\}_\Lambda$, we have a canonical isomorphism of left S-semimodules*

$$\mathrm{Hom}_T(F, \prod_{\lambda \in \Lambda} Y_\lambda) \simeq \prod_{\lambda \in \Lambda} \mathrm{Hom}_T(F, Y_\lambda).$$

(b) *For any inverse system of left T-semimodules $(X_j, \{f_{jj'}\})_J$, we have an isomorphism of left S-semimodules*

$$\mathrm{Hom}_T(F, \varprojlim X_j) \simeq \varprojlim \mathrm{Hom}_T(F, X_j).$$

(c) $\mathrm{Hom}_T(F, -)$ *preserves equalizers;*

(d) $\mathrm{Hom}_T(F, -)$ *preserves kernels.*

(3) $\mathrm{Hom}_T(-, F) : {}_T\mathbf{SM} \longrightarrow \mathbf{SM}_S$ *preserves all limits.*

(a) For every family of left T-semimodules $\{Y_\lambda\}_\Lambda$, we have a canonical isomorphism of right S-semimodules

$$\mathrm{Hom}_T(\bigoplus_{\lambda\in\Lambda} Y_\lambda, F,) \simeq \prod_{\lambda\in\Lambda} \mathrm{Hom}_T(Y_\lambda, F).$$

(b) For any directed system of left T-semimodules $(X_j, \{f_{jj'}\})_J$, we have an isomorphism of right S-semimodules

$$\mathrm{Hom}_T(\varinjlim X_j, F) \simeq \varprojlim \mathrm{Hom}_T(X_j, F).$$

(c) $\mathrm{Hom}_T(-, F)$ *converts coequalizers into equalizers;*

(d) $\mathrm{Hom}_T(-, F)$ *converts cokernels into kernels.*

Proof. The proof can be obtained as a direct consequence of Proposition 1.36 and the fact that $(F \otimes_S -, \mathrm{Hom}_T(F, -))$ is an adjoint pair of covariant functors [26].■

1.38 *An S-linear map $h : M \to N$ is called a* ***equalizer*** *of $f, g : N \to L$ if $f \circ h = g \circ h$ and whenever an S-linear map $h' : M' \to N$ satisfies $f \circ h' = g \circ h'$, there exists a unique S-linear map $\varphi : M' \to M$ such that $h \circ \varphi = h'$*

$$\begin{tikzcd}
M \arrow[r, "h"] & N \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & L & N \\
M' \arrow[u, dotted, "\exists!\ \varphi"] \arrow[ur, "h'"'] & & &
\end{tikzcd}$$

1.39 *An S-linear map $h : M \to N$ is called a* ***coequalizer*** *of $f, g : L \to M$ if $h \circ f = h \circ g$ and whenever an S-linear map $h' : M \to N'$ satisfies $h' \circ f = h' \circ g$, there exists a unique S-linear map $\varphi : N \to N'$ such that $\varphi \circ h = h'$*

$$\begin{tikzcd}
L \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & M \arrow[r, "h"] \arrow[dr, "h'"'] & N \arrow[d, dotted, "\exists!\ \varphi"] \\
& & N'
\end{tikzcd}$$

1.40 *Let $f : M \to N$ be an S-linear map. $Ker(f) := \{m \in M |\ f(m) = 0\}$. The map $ker(f) : Ker(f) \to M$ is the equalizer of f and the zero map. $Coker(f) := N/f(M)$. The map $coker(f) : N \to Coker(f)$ is the coequalizer of f and the zero map.*

Corollary 1.37 allows us to improve [38, Theorem 2.6].

Proposition 1.41 *Let ${}_TG_S$ be (T, S)-bisemimodule and consider the covariant functor* $\mathrm{Hom}_T(G, -) : {}_T\mathbf{SM} \longrightarrow {}_S\mathbf{SM}$. *Let*

$$0 \longrightarrow L \overset{f}{\to} M \overset{g}{\to} N \tag{1.10}$$

be a sequence of left T-semimodules and consider the following sequence of left S-semimodules

$$0 \longrightarrow \mathrm{Hom}_T(G, L) \overset{(G,f)}{\to} \mathrm{Hom}_T(G, M) \overset{(G,g)}{\longrightarrow} \mathrm{Hom}_T(G, N). \tag{1.11}$$

(1) *If the sequence* $0 \longrightarrow L \stackrel{f}{\to} M$ *is exact and* f *is normal, then*

$$0 \longrightarrow \mathrm{Hom}_T(G,L) \stackrel{(G,f)}{\to} \mathrm{Hom}_T(G,M)$$

is exact and (G,f) *is normal.*

(2) *If* (*1.10*) *is semi-exact and* f *is normal, then* (*1.11*) *is semi-exact* (*proper exact*) *and* (G,f) *is normal.*

(3) *If* (*1.10*) *is exact and* $\mathrm{Hom}_T(G,-)$ *preserves* k*-normal morphisms, then* (*1.11*) *is exact.*

Proof.

(1) The following implications are obvious: $0 \longrightarrow L \stackrel{f}{\to} M$ is exact $\Longrightarrow f$ is injective $\Longrightarrow (G,f)$ is injective $\Longrightarrow 0 \longrightarrow \mathrm{Hom}_T(G,L) \stackrel{(G,f)}{\to} \mathrm{Hom}_T(G,M)$ is exact. Assume that f is normal and consider the short exact sequence of S-semimodules

$$0 \longrightarrow L \stackrel{f}{\longrightarrow} M \stackrel{\pi_L}{\longrightarrow} M/L \longrightarrow 0.$$

Notice that $L = \mathrm{Ker}(\pi_L)$ by Lemma 1.29. By Corollary 1.37, $\mathrm{Hom}_T(G,-)$ preserves kernels and so $(G,f) = \ker(G,\pi_L)$ whence normal.

(2) Apply Lemma 1.29 (3): The semi-exactness of (*1.10*) and the normality of f are equivalent to $L \simeq \mathrm{Ker}(g)$. Since $\mathrm{Hom}_T(G,-)$ preserves kernels, we deduce that $\mathrm{Hom}_T(G,L) = \mathrm{Ker}((G,g))$ which is equivalent to the semi-

exactness of (*1.11*) and the normality of (G, f). Notice that

$$(G,f)(\mathrm{Hom}_T(G,L)) = \overline{(G,f)(\mathrm{Hom}_T(G,L))} = \mathrm{Ker}(G,g),$$

i.e. (*1.11*) is proper exact (whence semi-exact).

(3) The statement follows directly from (2) and the assumption on $\mathrm{Hom}_T(G,-)$. ■

Proposition 1.42 *Let ${}_TG_S$ be a (T,S)-bisemimodule and consider the functor* $\mathrm{Hom}_T(-,G) : {}_T\mathbf{SM} \longrightarrow \mathbf{SM}_S$. *Let*

$$L \stackrel{f}{\to} M \stackrel{g}{\to} N \longrightarrow 0 \tag{1.12}$$

be a sequence of left T-semimodules and consider the sequence of right S-semimodules

$$0 \longrightarrow \mathrm{Hom}_T(N,G) \stackrel{(g,G)}{\to} \mathrm{Hom}_T(M,G) \stackrel{(f,G)}{\longrightarrow} \mathrm{Hom}_T(L,G). \tag{1.13}$$

(1) *If $M \stackrel{g}{\to} N \longrightarrow 0$ is exact and g is normal, then $0 \longrightarrow \mathrm{Hom}_T(N,G) \stackrel{(g,G)}{\to}$ $\mathrm{Hom}_T(M,G)$ is exact and (g,G) is normal.*

(2) *If* (1.12) *is semi-exact and g is normal, then* (1.13) *is semi-exact (proper-exact) and (g,G) is normal.*

(3) *If* (1.12) *is exact and $\mathrm{Hom}_T(-,G)$ converts i-normal morphisms into k-normal ones, then* (1.13) *is exact.*

Proof.

(1) The following implications are clear: $M \xrightarrow{g} N \longrightarrow 0$ is exact $\Longrightarrow$ g is surjective $\Longrightarrow$ (g, G) is injective $\Longrightarrow$ $0 \longrightarrow \mathrm{Hom}_T(N, G) \xrightarrow{(g,G)} \mathrm{Hom}_T(M, G)$ is exact. Assume that g is normal and consider the exact sequence of S-semimodules

$$0 \longrightarrow \mathrm{Ker}(g) \xrightarrow{\iota} M \xrightarrow{g} N \longrightarrow 0.$$

Notice that $N \simeq \mathrm{Coker}(\iota)$. By Corollary 1.37, $\mathrm{Hom}_T(-, G)$ converts cokernels into kernels, we conclude that $(g, G) = \ker((f, G))$ whence normal.

(2) Apply Lemma 1.29 (5): $L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is semi-exact and g is normal $\Longleftrightarrow$ $M \simeq \mathrm{Coker}(f)$. Since the contravariant functor $\mathrm{Hom}_T(-, G)$ converts cokernels into kernels, it follows that $\mathrm{Hom}_T(N, G) = \mathrm{Ker}((f, G))$ which is in turn equivalent to (*1.13*) being semi-exact and (g, G) being normal. Notice that

$$(g, G)(\mathrm{Hom}_S(N, G)) = \overline{(g, G)(\mathrm{Hom}_S(N, G))} = \mathrm{Ker}((f, G)),$$

i.e. (*1.13*) is proper-exact (whence semi-exact).

(3) This follows immediately from "2" and the assumption on $\mathrm{Hom}_T(-, G)$. ■

Proposition 1.43 *Let* ${}_T G_S$ *be a* (T, S)*-bisemimodule and consider the functor* $G \otimes_S - : {}_S\mathbf{SM} \longrightarrow {}_T\mathbf{SM}$. *Let*

$$L \xrightarrow{f} M \xrightarrow{g} N \to 0 \tag{1.14}$$

be a sequence of left S-semimodules and consider the sequence of left T-semimodules

$$G \otimes_S L \overset{G\otimes f}{\longrightarrow} G \otimes_S M \overset{G\otimes g}{\longrightarrow} G \otimes_S N \to 0 \qquad (1.15)$$

(1) *If $M \overset{g}{\to} N \to 0$ is exact and g is normal, then $G \otimes_S M \overset{G\otimes g}{\longrightarrow} G \otimes_S N \to 0$ is exact and $G \otimes g$ is normal.*

(2) *If* (1.14) *is semi-exact and g is normal, then* (1.15) *is semi-exact and $G \otimes g$ is normal.*

(3) *If* (1.14) *is exact and $G \otimes_S -$ preserves i-normal morphisms, then* (1.15) *is exact.*

Proof. The following implications are obvious: $M \overset{g}{\to} N \to 0$ is exact $\Longrightarrow g$ is surjective $\Longrightarrow G \otimes g$ is surjective $\Longrightarrow G \otimes_S M \overset{G\otimes g}{\longrightarrow} G \otimes_S N \to 0$ is exact.

(1) Assume that g is normal and consider the exact sequence of S-semimodules

$$0 \longrightarrow \mathrm{Ker}(g) \overset{\iota}{\longrightarrow} M \overset{g}{\longrightarrow} N \longrightarrow 0.$$

Then $N \simeq \mathrm{Coker}(\iota)$. By Corollary 1.37 (1), $G \otimes_S -$ preserves cokernels and so $G \otimes g = \mathrm{coker}(G \otimes \iota)$ whence normal.

(2) Apply Lemma 1.29: The assumptions on (1.14) are equivalent to $N = \mathrm{Coker}(f)$. Since $G \otimes_S -$ preserves cokernels, we conclude that $G \otimes_S N = \mathrm{Coker}(G \otimes f)$, *i.e.* (1.15) is semi-exact and $G \otimes g$ is normal.

(3) This follows directly form (2) and the assumption on $G \otimes_S -$. ■

1.44 *Let $\gamma : T \longrightarrow S$ be a morphism of semirings. Then we have an* adjoint pair of functors $(F(X), Hom_T(S, -))$, *where $F(X) = X$ with the action $tx = \gamma(t)x$ for all $t \in T$ and $x \in X$ and $(s_1 f)(s) = f(ss_1)$ for all $s_1, s \in S$ and $f \in Hom_T(S, Y)$ for every left T-semimodule Y. In particular, we have for all $X \in {}_S\mathbf{SM}$ and $Y \in {}_T\mathbf{SM}$ a natural* isomorphism *of commutative monoids*

$$\theta_{X,Y} : Hom_S(X, Hom_T(S_S, Y)) \longrightarrow Hom_T(X, Y), \;\; f \mapsto [x \mapsto f(x)(1_S)] \quad (1.16)$$

with inverse

$$\phi_{X,Y} : Hom_T(X, Y) \longrightarrow Hom_S(X, Hom_T(S_S, Y)), \;\; g \mapsto [x \mapsto [s \mapsto g(sx)]]. \quad (1.17)$$

1.3 Pullbacks and Pushouts

Throughout, $(S, +, 0, \cdot, 1)$ is a semiring and, unless otherwise explicitly mentioned, an S-module is a **left** S-semimodule. The category of left S-semimodules is denoted by ${}_S\mathbf{SM}$.

The category ${}_S\mathbf{SM}$ of left S-semimodules has pullbacks and pushouts.

The pullbacks in ${}_S\mathbf{SM}$ are constructed in a way similar to that of pullbacks in category of modules over a ring.

1.45 *[35, 1.7] Let $f : A \to C$ and $g : B \to C$ be morphisms of left S-*

semimodules. The ***pullback*** *of* (f,g) *is* $(Q; f', g')$, *where*

$$\begin{aligned} Q\ &:\ = \{(a,b) \in A \times B \mid f(a) = g(b)\} \\ g'\ &:\ Q \to A,\ (a,b) \mapsto a; \\ f'\ &:\ Q \to C,\ (a,b) \mapsto b, \end{aligned} \tag{1.18}$$

$$\begin{tikzcd} Q^* \arrow[drr, bend left, "g^*"] \arrow[ddr, bend right, "f^*"'] \arrow[dr, dashed, "\varphi"] & & \\ & Q \arrow[r, "g'"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ & B \arrow[r, "g"'] & C \end{tikzcd} \tag{1.19}$$

and whenever $(Q^*; f^*, g^*)$ *satisfies* $f^* \circ g = g^* \circ f$, *there exists uniquely an* S*-linear map* $\varphi : Q^* \to Q$ *such that* $f \circ \varphi = f^*$ *and* $g \circ \varphi = g^*$.

Although the existence of pushouts in the category ${}_S\mathbf{SM}$ is guaranteed since this category is a variety (in the sense of Universal Algebra 1.16), the construction of pushouts in it is much more subtle that the construction of pushouts in the category of modules over a ring (mainly because of the lack of subtraction). This made some authors consider a special version of pushouts, e.g., Takahashi [35] who constructed in the so called C-pushouts, which coincide with the pushout in the category of cancellative semimodules.

1.46 *([35, 1.8]) Let* $f : L \to M$ *and* $g : L \to N$ *be morphisms of left* S*-*

semimodules. Consider the congruence $\sim$ *on* $M \oplus N$ *defined as*

$$(m_1, n_1) \sim (m_2, n_2) \Leftrightarrow \ \exists\ l_1, l_2 \in L : \ m_1 + f(l_1) = m_2 + f(l_2) \text{ and } n_1 + g(l_2) = n_2 + g(l_1). \tag{1.20}$$

The C-pushout *of* (f, g) *is*

$$\begin{aligned} CP \ &: \ = (\iota_M, \iota_N; (M \oplus N)/\sim); \\ \iota_M \ &: \ M \to CP, \ m \mapsto [(m, 0)]; \\ \iota_N \ &: \ N \to CP, \ n \mapsto [(0, n)]. \end{aligned} \tag{1.21}$$

While the C-pushouts coincide with the natural pushout in the subcategory ${}_S\mathbf{CSM}$ of cancellative left semimodules, they fail to have the *universal property* of pushouts in ${}_S\mathbf{SM}$.

In what follows, we demonstrate the construction of pushouts in S-semimodules ${}_S\mathbf{SM}$. The constructive proof is the objective of the following theorem which is already known to be true.

Theorem 1.47 *Let* $f : L \to M$ *and* $g : L \to N$ *be morphisms of left* S-*semimodules. Then* (f, g) *has a pushout.*

Proof. *Consider*

$$\begin{aligned} \mathcal{P} := \ &\{(g', f', P) \mid P \in \ {}_S\mathbf{SM}, \ g' : M \to P, f' : N \to P, \ g' \circ f = f' \circ g, \\ &\pi_{(g', f')} : M \oplus N \longrightarrow P, \ (m, n) \mapsto g'(m) + f'(n) \text{ is surjective}\}. \end{aligned}$$

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[d, "g'"] & \\ N \arrow[r, "f'"'] & P & M \oplus N \arrow[l, "\pi_{(g',f')}"] \end{tikzcd}$$

Notice that $\mathcal{P}$ is not empty as $(0,0,0) \in \mathcal{P}$.

Define a relation $\leq$ on $\mathcal{P}$ as $(\tilde{g}, \tilde{f}, U) \leq (f', g', P)$ if there exists an S-linear map $\alpha : P \to U$ such that $\alpha \circ \pi_{(g',f')} = \pi_{(\tilde{g},\tilde{f})}$, i.e. the following diagram is commutative

$$\begin{tikzcd} M \oplus N \arrow[r, "\pi_{(g',f')}"] \arrow[d, "\pi_{(\tilde{g},\tilde{f})}"'] & P \arrow[dl, dashed, "\alpha"] \\ U & \end{tikzcd}$$

Claim I: *$\mathcal{P}$ has a largest element $(\pi_M, \pi_N, \mathbf{P})$, where*

$$\begin{aligned} \mathbf{P} \quad &: \quad = (M \oplus N)/\rho, \\ (m_1, n_1)\rho(m_2, n_2) \quad &\Leftrightarrow \quad g_\lambda(m_1) + f_\lambda(n_1) = g_\lambda(m_2) + f_\lambda(n_2) \ \forall (g_\lambda, f_\lambda, P_\lambda) \in \mathcal{P}; \\ \pi_M \quad &: \quad M \longrightarrow (M \oplus N)/\rho, \ m \mapsto [(m, 0)]; \\ \pi_N \quad &: \quad M \longrightarrow (M \oplus N)/\rho, \ n \mapsto [(0, n)]. \end{aligned}$$

- *Notice that $(\pi_M, \pi_N, \mathbf{P}) \in \mathcal{P}$: for any $l \in L$, we have for any $(g_\lambda, f_\lambda, P_\lambda) \in \mathcal{P}$:*

$$(g_\lambda \circ f)(l) + f_\lambda(0_N) = (g_\lambda \circ f)(l) = (f_\lambda \circ g)(l) = g_\lambda(0_M) + (f_\lambda \circ g)(l)$$

whence (by the definition of ρ):

$$(\pi_M \circ f)(l) = [(f(l), 0)]_\rho = [(0, g(l))]_\rho = (\pi_N \circ g)(l).$$

- *For every $(g_\lambda, f_\lambda, P_\lambda) \in \mathcal{P}$, consider the S-linear map*

$$\alpha_\lambda : \mathbf{P} \longrightarrow P_\lambda,\ [(m, n)]_\rho \mapsto g_\lambda(m) + f_\lambda(n)$$

Notice that α_λ is well defined: If $[(m_1, n_1)]_\rho = [(m_2, n_2)]_\rho$, then it follows by the definition of ρ that

$$(\alpha_\lambda \circ \pi_{(\pi_M, \pi_N)})(m, n) = g_\lambda(m_1) + f_\lambda(n_1) = g_\lambda(m_2) + f_\lambda(n_2) = \alpha([(m_2, n_2)]_\rho).$$

Moreover, the following diagram

$$\begin{tikzcd} M \oplus N \arrow[r, "\pi_{(\pi_M,\pi_N)}"] \arrow[d, "\pi_{(g_\lambda, f_\lambda)}"'] & \mathbf{P} \arrow[dl, dashed, "\alpha_\lambda"] \\ P_\lambda & \end{tikzcd}$$

is commutative: indeed, for all $(m, n) \in M \oplus N$ we have

$$(\alpha_\lambda \circ \pi_{(\pi_M, \pi_N)})((m, n)) = \alpha_\lambda[(m, n)]_\rho = g_\lambda(m) + f_\lambda(n) = \pi_{(g_\lambda, f_\lambda)}(m, n).$$

Claim II: *A largest element $(g', f'; P)$ of $\mathcal{P}$ is a pushout of (f, g). By the definition of $\mathcal{P}$, we have $g' \circ f = f' \circ g$. So it remains to prove the it has the* *universal property of pushouts.*

- *Let Q be a left S-semimodule along with S-linear maps $g^* : M \to Q$ and $f^* : N \to Q$ satisfying $g^* \circ f = f^* \circ g$. Since $\pi_{(g',f')}$ is surjective, for each $p \in P$ there exists $(m,n) \in M \oplus N$, such that $p = g'(m) + f'(n)$. Define*

$$\varphi : P \to Q,\ p \mapsto g^*(m) + f^*(n).$$

$$\begin{tikzcd}
L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[d, "g'"] \arrow[ddr, bend left, "g^*"] & \\
N \arrow[r, "f'"'] \arrow[drr, bend right, "f^*"'] & P \arrow[dr, dashed, "\varphi"] & \\
& & Q
\end{tikzcd} \tag{1.22}$$

- *It follows directly from the definition of φ that $\varphi \circ g' = g^*$ and $\varphi \circ f' = f^*$. We prove that φ is* well defined. *Suppose that there exist (m_1, n_1), $(m_2, n_2) \in M \oplus N$ such that $g'(m_1) + f'(n_1) = p = g'(m_2) + f'(n_2)$.*

  Consider the equivalence on $M \oplus N$ defined by

$$(m,n)\omega(m',n') \text{ if } g^*(m) + f^*(n) = g^*(m') + f^*(n').$$

  Clearly, ω is a congruence. Let

$$\pi_M^\omega : M \longrightarrow (M \oplus N)/\omega, \pi_N^\omega : N \longrightarrow (M \oplus N)/\omega$$

be the canonical S-linear maps, and define

$$\pi_\omega : M \oplus N \to (M \oplus N)/\omega,\ (m,n) \mapsto [(m,n)]_\omega;$$
$$h : (M \oplus N)/\omega \longrightarrow Q,\ [(m,n)] \mapsto g^*(m) + f^*(n).$$

Notice that h is well defined by the definition of ω. Then $(\pi_M^\omega, \pi_N^\omega, (M \oplus N)/\omega) \in \mathcal{P}$. Since (g', f', P) is, by assumption, a largest element in $\mathcal{P}$, there exists $\alpha : P \to (M \oplus N)/\omega$ such that $\alpha \circ \pi_{(g',f')} = \pi_\omega$. It follows that

$$\begin{aligned}
\varphi(g'(m_1) + f'(n_1)) &= g^*(m_1) + f^*(n_1) &&= h([(m_1,n_1)]_\omega) \\
&= (h \circ \pi_\omega)(m_1,n_1) &&= (\alpha \circ \pi_{(g',f')})(m_1,n_1) \\
&= \alpha(g'(m_1) + f'(n_1)) &&= \alpha(g'(m_2) + f'(n_2)) \\
&= (\alpha \circ \pi_{(g',f')})(m_2,n_2) &&= (h \circ \pi_\omega)(m_2,n_2) \\
&= h([(m_2,n_2)]_\omega) &&= g^*(m_2) + f^*(n_2) \\
&= \varphi(g'(m_2) + f'(n_2)).
\end{aligned}$$

Hence φ is well defined.■

Corollary 1.48 *Let $f : L \to M$ and $g : L \to N$ be morphisms of left S-semimodules. There exists a congruence relation ρ on $M \oplus N$ such that*

$$(g', f'; (M \oplus N)/\rho),\ g'(m) := [(m,0)]_\rho,\ f'(n) := [(0,n)]_\rho$$

is a pushout of (f,g).

Proof. *Let (g^*, f^*, P) be a largest element in the poset $(\mathcal{P}, \leq)$ in the proof of*

Theorem 1.47. Then $(g^*, f^*; P)$ is a pushout and there is an surjective map

$$\pi : M \oplus N \longrightarrow P,\ (m, n) \mapsto g^*(m) + f^*(n).$$

Consider the congruence relation $\rho :=\equiv_\pi$ and define

$$\begin{aligned} g' &: M \to (M \oplus N)/\rho,\ m \mapsto [(m, 0)]_\rho \\ f' &: N \to (M \oplus N)/\rho,\ n) \mapsto [(0, n)]_\rho. \end{aligned}$$

For every $l \in L$, we have

$$(g' \circ f)(l) = [(f(l), 0)]_\rho = [(0, g(l))]_\rho = (f' \circ g)(l).$$

The middle equality follows since

$$\begin{aligned} \pi((f(l), 0)) &= (g^* \circ f)(l) + f^*(0) &= (f^* \circ g)(l) + 0 \\ &= g^*(0) + (f^* \circ g)(l) &= \pi((0, g(l))) \end{aligned}$$

With the canonical map $\pi_\rho : M \oplus N \to (M \oplus N)/\rho$, we have $(g', f', (M \oplus N)/\rho) \in \mathcal{P}$. Moreover $P \leq (M \oplus N)/\rho$ noticing that

$$\alpha : (M \oplus N)/\rho \longrightarrow P,\ [m, n]_\rho \mapsto g^*(m) + f^*(n)$$

is S-linear such that $\alpha\pi_\rho = \pi_{(g^*, f^*)}$. Since (g^*, f^*, P) is a largest element in $\mathcal{P}$, $(g', f', (M \oplus N)/\rho)$ is also a largest element in $\mathcal{P}$. Thus $(g', f'; (M \oplus N)/\rho)$ is a *pushout of* (f, g). ■

Lemma 1.49 *Let* $(g', f'; P)$ *be a pushout of the morphisms of left* S*-semimodules* $f : L \to M$ *and* $g : L \to N$.

(1) If f *is surjective, then* f' *is surjective.*

(2) If $f(L) \subseteq M$ *is subtractive, then* $f'(N) \subseteq P$ *is subtractive.*

(3) If f *is a normal epimorphism, then* f' *is a normal epimorphism (cf. [10, Proposition 2.2]).*

(4) If f *is injective and* g *is a normal epimorphism, then* f' *is injective.*

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[d, "g'"] \\ N \arrow[r, "f'"] & P \end{tikzcd}$$

Proof. *Let* $(g', f'; P)$ *be a pushout of* (f, g).

(1) *Let* $p \in P$. *Since* $\pi_{(g',f')}$ *is surjective, there exists* $(m, n) \in M \oplus N$ *such that* $p = \pi_{(g',f')}(m, n) = g'(m) + f'(n)$. *Since* f *surjective, there exists* $l \in L$ *such that* $f(l) = m$. *Consider* $g(l) + n \in N$. *It follows that*

$$f'(g(l) + n) = (f' \circ g)(l) + f'(n) = (g' \circ f)(l) + f'(n) = g'(m) + f'(n) = p.$$

(2) *Let* $p \in P$ *be such that* $p + f'(n_1) = f'(n_2)$ *for some* $n_1, n_2 \in N$. *Pick* $(m, n) \in M \oplus N$ *such that* $p = \pi_{(g',f')}(m, n) = g'(m) + f'(n)$. *Thus* $g'(m) +$

$f'(n+n_1) = f'(n_2)$.

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[d, "g'"] \arrow[ddr, bend left, "g^*"] & \\ N \arrow[r, "f'"] \arrow[drr, bend right, "f^*"'] & P \arrow[dr, "\varphi"] & \\ & & Q \end{tikzcd}$$

Let φ be the map from P to the C-pushout Q such that $\varphi \circ g' = g^*$ and $\varphi \circ f' = f^*$. Then

$$[(m, n+n_1)]_\sim = \varphi(g'(m) + f'(n+n_1)) = \varphi(f'(n_2)) = [(0, n_2)]_\sim.$$

By the definition of the congruence relation $\sim$ (1.20), there exist $l_1, l_2 \in L$ such that $m + f(l_1) = f(l_2)$ and $n + n_1 + g(l_2) = n_2 + g(l_1)$. Since $f(L) \subseteq M$ is subtractive, $m = f(l)$ for some $l \in L$. Then we have

$$p = g'(m) + f'(n) = (g' \circ f)(l) + f'(n) = (f' \circ g)(l) + f'(n) = f'(g(l) + n).$$

It follows that $f'(N) \subseteq P$ is subtractive.

(3) *Without loss of generality, let the pushout be $P = (g', f'; (M \oplus N)/\rho)$ for some congruence relation ρ on $M \oplus N$ and g', f' are the canonical maps (see Corollary 1.48). Since f is surjective, it follows by (1) that f' is surjective as well.*

Step I: *Consider the canonical S-linear map*

$$f^* : N \to N/Ker(f').$$

Let $m \in M$ and pick $l \in L$ such that $m = f(l)$. Define

$$g^* : M \to N/Ker(f'),\ m \mapsto (f^* \circ g)(l).$$

We prove that g^* is well-defined.

Suppose that $f(l) = m = f(l')$ for some $l, l' \in L$. Since f is k-normal, there exist $l_1,\ l_2 \in Ker(f)$ such that $l + l_1 = l' + l_2$. It follows that $g(l) + g(l_1) = g(l + l_1) = g(l' + l_2) = g(l') + g(l_2)$ with $(f' \circ g)(l_1) = (g' \circ f)(l_1) = 0 = (g' \circ f)(l_2) = (f' \circ g)(l_2)$. Thus $(f^* \circ g)(l) = [g(l)]_{Ker(f')} = [g(l')]_{Ker(f')} = (f^* \circ g)(l')$. Clearly, g^* is S-linear and satisfies $g^* \circ f = f^* \circ g(l)$.

Step II: *Define*

$$\psi : N/Ker(f') \to P,\ [n]_{Ker(f')} \mapsto [(0, n)]_\rho.$$

$$\begin{tikzcd}
L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[d, "g'"] \arrow[ddr, bend left, "g^*"] & \\
N \arrow[r, "f'"] \arrow[drr, bend right, "f^*"'] & P & \\
& & N/Ker(f') \arrow[ul, "\psi"']
\end{tikzcd}$$

Notice that ψ is well defined: Suppose that $[n]_{Ker(f')} = [n']_{Ker(f')}$ for some

$n, n' \in N$. It follows that $n + n_1 = n + n_2$ for some $n_1, n_2 \in Ker(f')$. Thus

$$\begin{aligned} [(0,n)]_\rho &= [(0,n)]_\rho + [(0,0)]_\rho &= [(0,n)]_\rho + f'(n_1) \\ &= [(0,n)]_\rho + [(0,n_1)]_\rho &= [(0,n+n_1)]_\rho \\ &= [(0,n'+n_2)]_\rho &= [(0,n')]_\rho. \end{aligned}$$

For $m \in M$ pick some $l \in L$ with $f(l) = m$. Then we have

$$\begin{aligned} (\psi \circ g^*)(m) &= (\psi \circ f^* \circ g)(l) &= \psi([g(l)]_{Ker(f')}) \\ &= [(0, g(l))]_\rho &= (f' \circ g)(l) \\ &= (g' \circ f)(l) &= g'(m), \end{aligned}$$

whence $\psi \circ g^* = g'$.

On the other hand, for every $n \in N$ we have $(\psi \circ f^*)(n) = \psi([n]_{Ker(f')}) = [(0,n)]_\rho = f'(n)$, whence $(\psi \circ f^*) = f'$.

Step III: *Since P is a pushout, there exists an S-linear map $\varphi : P \to N/Ker(f')$ such that $\varphi \circ g' = g^*$ and $\varphi \circ f' = f^*$. For each $(m,n) \in M \oplus N$ we have*

$$\begin{aligned} (\psi \circ \varphi)([(m,n)]_\rho) &= \psi(\varphi([(m,0)]_\rho) + \varphi([(0,n)]_\rho) \\ &= \psi((\varphi \circ g')(m) + (\varphi \circ f')(n)) \\ &= (\psi \circ g^*)(m) + (\psi \circ f^*)(n)) \\ &= g'(m) + f'(n) \\ &= [(m,n)]_\rho. \end{aligned}$$

On the other hand, we have for every $n \in N$:

$$(\varphi \circ \psi)([n]_{Ker(f')}) = \varphi[(0,n)]_\rho = (\varphi \circ f')(n) = f^*(n) = [n]_{Ker(f')}.$$

Hence $P \simeq N/Ker(f')$. This implies that f' is k-normal (as f^* is obviously k-normal).

(4) *Without loss of generality, let the pushout be $P = (g', f'; (M \oplus N)/\rho)$ for some congruence relation ρ on $M \oplus N$ and g', f' are the canonical maps (see Corollary 1.48). Let $K := f(Ker(g))$ and consider the canonical projection $\tilde{g} : M \to M/K$. By assumption, g is surjective and so there exists for every $n \in N$ some $l_n \in L$ such that $n = g(l_n)$.*

Step I: *Define*

$$\tilde{f} : N \to M/K,\ n \mapsto [f(l_n)]_K.$$

We claim that $\tilde{f}$ is well defined. Suppose that $g(l_n) = n = g(l'_n)$. Since g is k-normal, there exist $l_1, l_2 \in Ker(g)$ such that $l_n + l_1 = l'_n + l_2$, whence $f(l_n) + f(l_1) = f(l'_n) + f(l_2)$, i.e. $[f(l_n)]_K = [f(l'_n)]_K$ (recall that we chose $K := f(Ker(g))$).

$$\begin{tikzcd}
L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[d, "g'"] \arrow[ddr, bend left, "\tilde{g}"] & \\
N \arrow[r, "f'"] \arrow[drr, bend right, "\tilde{f}"'] & P \arrow[dr, "\varphi"] & \\
 & & M/K
\end{tikzcd}$$

Notice that for every $l \in L$, we have: $(\tilde{f} \circ g)(l) = [f(l)]_K = (\tilde{g} \circ f)(l)$.

Since P is a pushout, there exists an S-linear map $\varphi : P \to M/K$ such that $(\varphi \circ g') = \tilde{g}$ and $(\varphi \circ f') = \tilde{f}$.

Step II: *Define*

$$\psi : M/K \to P,\ [m]_K \mapsto [(m, 0)]_\rho.$$

We claim that ψ is well defined. *Suppose that $[m]_K = [m']_K$ for some $m, m' \in M$. Then there exist $l_1, l_2 \in Ker(g)$ such that $m + f(l_1) = m' + f(l_2)$. It follows that*

$$\begin{aligned}
[(m,0)]_\rho &= g'(m) &&= g'(m) + 0 \\
&= g'(m) + (f' \circ g)(l_1) &&= g'(m) + (g' \circ f)(l_1) \\
&= g'(m + f(l_1)) &&= g'(m' + f(l_2)) \\
&= [m', 0]_\rho.
\end{aligned}$$

Step III: *Notice that for every $n = f(l_n) \in N$ we have:*

$$\begin{aligned}
(\psi \circ \tilde{f})(n) &= \psi[f(l_n)]_K &&= [(f(l_n), 0)]_\rho \\
&= (g' \circ f)(l_n) &&= (f' \circ g)(l_n) \\
&= f'(n),\ \textit{and}
\end{aligned}$$

$$\begin{aligned}
(\psi \circ \tilde{g})(m) &= \psi([m]_K) &&= [(m, 0)]_\rho \\
&= g'(m),
\end{aligned}$$

thus $\psi \circ \tilde{f} = f'$ *and* $\psi \circ \tilde{g} = g'$. *Moreover,*

$$\begin{aligned} (\varphi \circ \psi)([m]_K) &= \varphi[(m,0)]_\rho = (\varphi \circ g')(m) \\ &= \tilde{g}(m) = [m]_K, \text{ and} \end{aligned}$$

$$\begin{aligned} (\psi \circ \varphi)([(m,0)]_\rho) &= (\psi \circ \varphi \circ g')(m) = (\psi \circ \tilde{g})(m) \\ &= \psi([m]_K) = [(m,0)]_\rho, \end{aligned}$$

i.e. ψ, φ *are* S*-linear isomorphisms and* $\psi^{-1} = \varphi$. *Moreover,* M/K *is a pushout.*

Step IV: *Let* $n, n' \in N$ *be such that* $\tilde{f}(n) = \tilde{f}(n')$, *i.e.* $[f(l_n)]_K = [f(l_{n'})]_K$. *Then there exist* $l_1, l_2 \in Ker(g)$ *such that* $f(l_n+l_1) = f(l_n)+f(l_1) = f(l_{n'})+f(l_2) = f(l_{n'}+l_2)$, *whence* $l_n + l_1 = l_{n'} + l_2$ *as* f *is injective. It follows that* $n = g(l_n) = g(l_n)+g(l_1) = g(l_n+l_1) = g(l_{n'}+l_2) = g(l_{n'})+g(l_2) = g(l_{n'}) = n'$. *Thus* $\tilde{f}$ *is injective. Since* $f' = \psi \circ \tilde{f}$ *and* $\psi, \tilde{f}$ *are injective,* f' *is injective as well.*$\blacksquare$

CHAPTER 2

PROJECTIVE, INJECTIVE AND FLAT SEMIMODULES

As before, $(S, +, 0, \cdot, 1)$ is a semiring and, unless otherwise explicitly mentioned, an S-module is a **left** S-semimodule. Exact sequences here are in the sense of Abuhlail [1] (see Definition 1.12).

2.1 Projective Semimodules

There are several notions of projectivity for a semimodule over a semiring, which coincide if it were a module over a ring. In this Chapter, we consider some of them and clarify the relationships between them, and then investigate the so called *e-projective semimodules* which turn to coincide with the so called *normally projective semimodules* (both notions introduced by Abuhlail [3, 1.25, 1.24] and called *uniformly projective semimodules*). The terminology "*e-projective*" appeared first in [6]).

Definition 2.1 *([6]) A left S-semimodules P is*

*M-e-**projective** (where M is a left S-semimodule) if the covariant functor*

$$Hom_S(P, -) : \ {}_S\mathbf{SM} \longrightarrow \ {}_{\mathbb{Z}^+}\mathbf{SM}$$

transfers every short exact sequence of left S-semimodules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

into a short exact sequence of commutative monoids

$$0 \longrightarrow Hom_S(P, L) \xrightarrow{(P,f)} Hom_S(P, M) \xrightarrow{(P,g)} Hom_S(P, N) \longrightarrow 0.$$

We say that P is e-***projective*** *if P is M-e-projective for every left S-semimodule M.*

2.1 *Let P be a left S-semimodule.*

For a left S-semimodule M, we say that P is

*M**-projective** [18, page 195] if for every* surjective *S-linear map $f : M \to N$ and an S-linear map $g : P \to N$, there exists an S-linear map $h : P \to M$ such that $h \circ g = f$;*

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r] & 0 \\ & P \arrow[u, "g"] \arrow[ul, dotted, "h"] & \end{tikzcd}$$

*M-k-**projective** [12, Definition 6] if for every* normal epimorphism *$f : M \to$*

N and any S-linear map $g : P \to N$, there exists an S-linear map $h : P \to M$ such that $h \circ g = f$;

$$\begin{tikzcd} M \arrow[r, "f(normal)"] & N \arrow[r] & 0 \\ & P \arrow[u, "g"'] \arrow[ul, dotted, "h"] & \end{tikzcd}$$

***normally** M-**projective** [3, 1.25] if for every* normal epimorphism *$f : M \to N$ and any S-linear map $g : P \to N$, there exists an S-linear map $h : P \to M$ such that $h \circ g = f$*

$$\begin{tikzcd} P \arrow[r, shift left, "h_2"] \arrow[r, shift right, "h_1"'] & M \arrow[r, "f(normal)"] & N \arrow[r] & 0 \\ & & P \arrow[u, "g"'] \arrow[ul, dotted, "h"] \arrow[ul, bend right, "h'"'] & \end{tikzcd}$$

and whenever an S-linear map $h' : P \to M$ satisfies $h \circ g = f$, there exist S-linear maps $h_1, h_2 : P \to M$ such that $f \circ h_1 = 0 = f \circ h_2$ and $h + h_1 = h' + h_2$.

We say that P is projective (resp., k-projective, normally projective) if P is M-projective (resp., M-k-projective, normally M-projective) for every left S-semimodule M.

Remarks 2.2 (1) *It is obvious from the definitions that projective and e-projective semimodules are k-projective.*

(2) *Despite being a retract of a free semimodule, a projective semimodule is not necessarily a direct summand of a free semimodule. For a counter example see ([10, Example 2.3]).*

Proposition 2.3 ([36, Theorem 1.9], [18, Proposition 17.16]) *A left S-semimodule ${}_SP$ is projective if and only if P is a retract of a free left S-semimodule.*

Proposition 2.4 *Let P be a left S-semimodule.*

(1) *${}_SP$ is M-e-projective (for some left S-semimodule M) if and only if ${}_SP$ is normally M-projective.*

(2) *${}_SP$ is e-projective if and only if ${}_SP$ is normally projective.*

Proof. We need to prove (1) only.

($\Longrightarrow$) Assume that ${}_SP$ is M-e-projective. Let $f : M \to N$ be a normal epimorphism and $g : P \to N$ an S-linear map. By Lemma 1.29, the sequence

$$0 \longrightarrow Ker(f) \xrightarrow{\iota} M \xrightarrow{f} N \longrightarrow 0$$

is a short exact sequence, where ι is the canonical embedding. By assumption, the following sequence of commutative monoids

$$0 \longrightarrow Hom_S(P, Ker(f)) \xrightarrow{(P,\iota)} Hom_S(P, M) \xrightarrow{(P,g)} Hom_S(P, N) \longrightarrow 0$$

is exact. In particular, (P, g) is surjective and k-normal, whence P is M-projective.

($\Longleftarrow$) let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be a short exact sequence of left S-semimodules and consider the induces sequences of commutative monoids

$$0 \longrightarrow Hom_S(P, L) \xrightarrow{(P,f)} Hom_S(P, M) \xrightarrow{(P,g)} Hom_S(P, N) \longrightarrow 0.$$

By Proposition 1.41, (P, f) is a normal monomorphism and $\mathrm{Im}((P, f)) = Ker((P, g))$. By assumption, (P, g) is a normal epimorphism, whence the induced sequence of commutative monoids is exact.$\blacksquare$

Following an observation by H. Al-Thani made in [9, theorem 4], we provide a detailed proof that every projective S-semimodule is e-projective.

Proposition 2.5 *Every projective left S-semimodule is e-projective.*

Proof. Let ${}_SP$ be projective. Assume that $M \xrightarrow{g} N \longrightarrow 0$ is a normal epimorphism of left S-semimodules, and $\alpha \in \mathrm{Hom}_S(P, N)$. Since ${}_SP$ is M-projective,

$$\mathrm{Hom}_S(P, M) \xrightarrow{(P,g)} \mathrm{Hom}_S(P, N) \longrightarrow 0$$

is surjective, *i.e.* there exists $\beta \in \mathrm{Hom}_S(P, M)$ such that $g \circ \beta = \alpha$.

By Proposition 2.4, it is enough to prove that (P, g) is k-normal.

Suppose that $(P, g)(\beta) = (P, g)(\beta')$ for some $\beta, \beta' \in \mathrm{Hom}_S(P, M)$, i.e. $g \circ \beta = g \circ \beta'$. Since ${}_SP$ is projective, P is a retract of a free left S-semimodule, *i.e.* there exists an index set Λ and a surjective S-linear map $\theta : S^{(\Lambda)} \longrightarrow P$ as well as an injective S-linear map $\psi : P \longrightarrow S^{(\Lambda)}$ such that $\theta \circ \psi = id_P$. Notice that $g \circ \beta \circ \theta = g \circ \beta' \circ \theta$. For every $\lambda \in \Lambda$, and since g is k-normal, there exist m_λ, $m'_\lambda \in \mathrm{Ker}(g)$ such that $(\beta \circ \theta)(\lambda) + m_\lambda = (\beta' \circ \theta)(\lambda) + m'_\lambda$. Let $\gamma, \gamma' \in \mathrm{Hom}_S(S^{(\Lambda)}, M)$ be the *unique* S-linear maps with $\gamma(\lambda) = m_\lambda$ and $\gamma'(\lambda) = m'_\lambda$ for each $\lambda \in \Lambda$ (they exist and are unique since Λ is a basis for $S^{(\Lambda)}$). It follows that

$$g \circ (\gamma \circ \psi) = (g \circ \gamma) \circ \psi = 0 = (g \circ \gamma') \circ \psi = g \circ (\gamma' \circ \psi),$$

i.e. $\gamma \circ \psi$, $\gamma' \circ \psi \in \mathrm{Ker}((P, g))$. Moreover, for any $\lambda \in \Lambda$ we have

$$(\beta \circ \theta + \gamma)(\lambda) = (\beta \circ \theta)(\lambda) + m_\lambda = (\beta' \circ \theta)(\lambda) + m'_\lambda = (\beta' \circ \theta + \gamma')(\lambda),$$

whence $\beta \circ \theta + \gamma = \beta' \circ \theta + \gamma'$. It follows that

$$\begin{aligned} \beta + \gamma \circ \psi &= \beta \circ id_P + \gamma \circ \psi &&= \beta \circ (\theta \circ \psi) + \gamma \circ \psi \\ &= (\beta \circ \theta + \gamma) \circ \psi &&= (\beta' \circ \theta + \gamma') \circ \psi \\ &= \beta' \circ (\theta \circ \psi) + \gamma' \circ \psi &&= \beta' \circ id_P + \gamma' \circ \psi \\ &= \beta' + \gamma' \circ \psi. \blacksquare \end{aligned}$$

The next example shows that the class of S-e-projective left S-e-projective left S-semimodules is strictly larger than that of S-projective left S-semimodules.

Example 2.6 *Consider the semiring $S := \mathbb{Q}^+$ of non-negative rational numbers, with the usual addition and multiplication. Consider the Boolean algebra $\mathbb{B}$ as an S-semimodule with $s \cdot 1 = 1 \Leftrightarrow s \in S \backslash \{0\}$. Then ${}_S\mathbb{B}$ is S-e-projective but not an S-projective S-semimodule.*

Proof. Consider the S-linear map

$$f : S \to \mathbb{B}, \; s \mapsto \left\{ \begin{array}{ccc} 1 & , & s \neq 0 \\ & & \\ 0, & & s = 0 \end{array} \right|$$

Notice that f is not k-normal: $Ker(f) = \{0\}$, $f(1) = 1 = f(2)$, and $1 + 0 \neq 2 + 0$.

Since there is no surjective S-linear map from $\mathbb{B}$ to S, there is no isomorphism from $\mathbb{B}$ to S. Since S is an ideal-simple S-semimodules, $Hom_S(\mathbb{B}, S) = \{0\}$ by Lemma 1.15. Since the following diagram

$$\begin{tikzcd} S \arrow[r, "f"] & \mathbb{B} \\ & \mathbb{B} \arrow[ul, "0"] \arrow[u, "id_{\mathbb{B}}"'] \end{tikzcd}$$

cannot be completed commutatively, B is not S-projective.

Let N be an S-semimodule and $f : S \to N$ be a normal S-epimorphism. If $f = 0$, then $N = f(S) = 0$, which implies every S-linear map $g : \mathbb{B} \to N$ is the zero morphism and by choosing S-linear map $0 = h : \mathbb{B} \to S$ we have $f = h \circ g$. If $f \neq 0$, then $f(1) \neq 0$. For every $s \in S \backslash 0$, we have $0 \neq f(1) = f(s^{-1}s) = s^{-1}f(s)$, whence $f(s) \neq 0$. Thus $Ker(f) = \{0\}$. If $f(s) = f(t)$, then $s + k_1 = t + k_2$ for some $k_1, k_2 \in Ker(f) = \{0\}$, thus $s = t$. Hence, f is an S-linear map and N is S-isomorphic to S. Since S is not S-isomorphic to $\mathbb{B}$, N is not S-isomorphic to $\mathbb{B}$. Since S is ideal-simple, N is ideal-simple. Thus $Hom_S(\mathbb{B}, N) = \{0\}$ and $\mathbb{B}$ is S-e-projective.■

2.7 *We say that a sequence of S-semimodules*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{2.1}$$

is

left splitting *if there exists* $f' \in \mathrm{Hom}_S(B, A)$ *such that* $f' \circ f = id_A$;

right splitting *if there exists* $g' \in \mathrm{Hom}_S(C, B)$ *such that* $g \circ g' = id_C$.

We say that (2.1) splits or is splitting if it is left splitting and right splitting.

A short exact sequence of modules over a ring is left splitting if and only if it is right splitting. However, this is not the case for semimodules over arbitrary semirings.

Example 2.8 *The short exact sequence of commutative monoids*

$$0 \longrightarrow \{0, 2\} \stackrel{\iota}{\longrightarrow} B(3, 1) \stackrel{\pi}{\longrightarrow} \mathbb{Z}_2 \longrightarrow 0, \tag{2.2}$$

where ι *is the canonical embedding and* π *is the canonical projection. The sequence (2.2) is exact since* $\{0, 2\}$ *is subtractive and* $B(3, 1)/\{0, 2\} \cong {}_{\mathbb{Z}^+}\mathbb{Z}_2$ *(see Lemma 1.29). Consider*

$$f : B(3, 1) \longrightarrow \{0, 2\}, \ x \mapsto \begin{cases} 2, & x \neq 0 \\ \\ 0, & x = 0 \end{cases}$$

and notice that $f \circ \iota = id_{\{0,2\}}$, *i.e. (2.2) is left splitting. On the other hand, we have* $Hom_{\mathbb{Z}^+}(\mathbb{Z}_2, B(3, 1)) = \{0\}$ *since* 1 *has an additive inverse (namely* 1*) in* $\mathbb{Z}_2$, *while no non-zero element of* $B(3, 1)$ *has an additive inverse. Consequently, (2.2) is not right splitting.*

Proposition 2.9 *A left* S*-semimodule* ${}_SP$ *is* k*-projective if and only if every short* *exact sequence of S-semimodules*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$$

is right-splitting.

Proof. ($\Rightarrow$) Let P be k-projective and $0 \to L \xrightarrow{f} M \xrightarrow{g} P \to 0$ be a short exact sequence. In particular, g is surjective and k-normal. Consider, $id_P : P \longrightarrow P$. Since ${}_SP$ is k-projective, there exists an S-linear map $g' : P \to M$ such that the following diagram

$$\begin{tikzcd} & P \arrow[dl, dotted, "g'"'] \arrow[d, "id_P"] \\ M \arrow[r, "g"'] & P \end{tikzcd}$$

is commutative, i.e. $g \circ g' = id_P$.

($\Leftarrow$) Let $M \xrightarrow{g} N \longrightarrow 0$ be a normal surjective S-linear map and $h : P \to N$ be a morphism of left S-semimodules. Consider the pullback of g and h :

$$Q = \{(p, m) \in P \times M \mid h(p) = g(m)\}$$

and the following commutative diagram

$$\begin{tikzcd}
Q \arrow[r, "\pi_P"] \arrow[d, "\pi_M"'] & P \arrow[d, "h"] \\
M \arrow[r, "g"'] & N
\end{tikzcd}$$

where π_P and π_Q are the canonical projections. Since g is surjective, $h(p) = g(m)$ for some $m \in M$, i.e. $(p, m) \in Q$ and indeed, $p = \pi_P(p, m)$. Hence π_P is surjective. Let $(p, m), (p, m') \in Q$ so that $\pi_P(p, m) = \pi_P(p, m')$. Then $g(m) = h(p) = g(m')$ and there exist $u, u' \in Ker(g)$ such that $m + u = m' + u'$ (since g is k-normal). Notice that $(0, u), (0, v) \in Ker(\pi_P)$ and $(p, m)+(0, u) = (p, m+u) = (p, m'+u') = (p, m) + (0 + u')$, i.e. π_P is k-normal. Hence the sequence

$$0 \to Ker(\pi_P) \hookrightarrow Q \xrightarrow{\pi_P} P \to 0$$

is exact, and there exists by our assumption an S-linear map $\varphi : P \to Q$ such that $\pi_P \circ \varphi = id_P$. Notice that for every $p \in P$, $\varphi(p) \in Q$ whence $\varphi(p) = (p, m)$ for some $m \in M$ with $h(p) = g(m)$. It follows that

$$(g \circ (\pi_M \circ \varphi))(p) = g(\pi_M(p, m)) = g(m) = h(p). \tag{2.3}$$

So, $g \circ (\pi_M \circ \varphi) = h$. Consequently, P is k-projective.■

Lemma 2.10 *If M is a left S-semimodule such that every subtractive subsemimodule is a direct summand, then every left S-semimodule is M-e-projective.*

Proof. *Let P be a left S-semimodule and let*

$$f : M \longrightarrow N \longrightarrow 0$$

be a normal epimorphism and $g : P \to N$ be an S-linear map. Notice that $Ker(f) \leq_S M$ is a subtractive subsemimodule, whence $M = Ker(f) \oplus L$ for some subsemimodule $L \leq_S M$. The row of this following diagram is exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Ker(f) & \xrightarrow{\iota} & M & \xrightarrow{f} & N & \longrightarrow & 0 \\ & & & & & & \uparrow{\scriptstyle g} & & \\ & & & & & & P & & \end{array}$$

by Lemma 1.29 and it follows (see also Remark 1.19(2)) that we have isomorphisms of left S-semimodules:

$$N \simeq M/Ker(f) \simeq L.$$

Considering the induced isomorphism $N \overset{g'}{\simeq} L$ and setting $h := \iota_L \circ g' \circ g : P \longrightarrow M$ where $f \circ \iota_L = id_L$ and $\iota_L \circ f|_L = id_N$, we have indeed $f \circ h = g$.

Suppose that also $h' : P \longrightarrow M$ satisfies $f \circ h' = g$. Consider the projection $\pi : M \longrightarrow Ker(f)$. Then

$$\varphi : M \longrightarrow M,\ m \mapsto \iota_L \circ g' \circ f + \pi$$

is the identity map: Let $m \in M$, and write $m = k+l$ for some unique $k \in Ker(f)$ and $l \in L$, and notice that

$$\varphi(m) = \varphi(k+l) = (\iota_L \circ g' \circ f + \pi)(k+l) + (\iota_L \circ g' \circ f)(k+l) + \pi(k+l) = l + k = m.$$

Choose $h_1 := \pi \circ h' : P \longrightarrow M$ and $h_2 = 0 : P \longrightarrow M$. Notice that $f \circ h_1 = f \circ \pi \circ h' = 0 = f \circ h_2$. Moreover, we have for each $p \in P$:

$$\begin{array}{rcccl} (h + h_1)(p) & = & h(p) + h_1(p) & = & (\iota_L \circ g' \circ g)(p) + (\pi \circ h')(p) \\ & = & (\iota_L \circ g' \circ f \circ h')(p) + \pi \circ h'(p) & = & ((\iota_L \circ g' \circ f + \pi) \circ h')(p) \\ & = & h'(p) & = & (h' + 0)(p). \end{array}$$

Consequently, P is M-e-projective.■

Lemma 2.11 *(cf. [6, Corollary 3.3])*

(1) *Let M be a left S-semimodule. A retract of an M-e-projective semimodule is M-e-projective.*

(2) *A retract of an e-projective left S-semimodule is e-projective.*

Proof. *We only need to prove (1).*

Let P be a left S-semimodule which is M-e-projective and let ${}_SK$ be a retract of P along with a surjective S-linear map $\pi_K : P \to K$ and an injective S-linear map $\iota_K : K \to P$ such that $\pi_K \circ \iota_K = id_K$.

Let $f : M \to N$ be a normal epimorphism and $g : K \to N$ an S-linear map.

Since P is e-projective, there exists an S-linear map $h^* : P \to M$ such that $f \circ h^* = g \circ \pi_K$. Consider $h := h^* \circ \iota_K : K \to M$.

$$\begin{tikzcd}
M \arrow[r, "f"] & N \arrow[r] & 0 \\
& K \arrow[u, "g"'] \arrow[d, shift right, "\iota_K"'] & \\
& P \arrow[u, shift right, "\pi_K"'] \arrow[uul, dashed, bend left, "h^*"] &
\end{tikzcd}$$

Then $f \circ h = f \circ (h^* \circ \iota_K) = g \circ \pi_K \circ \iota_K = g \circ id_K = g$.

Suppose that $h' : K \to M$ is an S-linear map such that $f \circ h' = g$. Since P is M-e-projective and $f \circ (h' \circ \pi_K) = (f \circ h') \circ \pi_K = g \circ \pi_K$, there exist S-linear maps $h'_1, h'_2 : P \to M$ such that $f \circ h'_1 = 0 = f \circ h'_2$ and $h^* + h'_1 = h' \circ \pi_K + h'_2$. Consider $h_1 := h'_1 \circ \iota_K$ and $h_2 := h'_2 \circ \iota_K$.

$$\begin{tikzcd}
K \arrow[r, "\iota_K"] & P \arrow[r, shift left, "h'_1"] \arrow[r, shift right, "h'_2"'] & M \arrow[r, "f"] & N \arrow[r] & 0 \\
& & & K \arrow[u, "g"'] \arrow[ul, bend right, "h"'] \arrow[ul, bend left, "h'"] \arrow[d, shift right, "\iota_K"'] & \\
& & & P \arrow[u, shift right, "\pi_K"'] \arrow[uul, bend left, "h^*"] &
\end{tikzcd}$$

Then $f \circ h_1 = f \circ h'_1 \circ \iota_K = 0$, $f \circ h_2 = f \circ h'_2 \circ \iota_K = 0$, and

$$\begin{aligned}
h + h_1 &= h^* \circ \iota_K + h'_1 \circ \iota_K &&= (h^* + h'_1) \circ \iota_K \\
&= (h' \circ \pi_K + h'_2) \circ \iota_K &&= h' \circ \pi_K \circ \iota_K + h'_2 \circ \iota_K \\
&= h' + h_2.
\end{aligned}$$

Consequently, K is M-e-projective. ■

Proposition 2.12 *Let* $\{P_i\}_{I\in I}$ *be a family of left* S*-semimodules and* M *a left* S*-semimodule. Then* $\bigoplus_{i\in I} P_i$ *is* M*-e-projective if and only if* P_i *is* M*-e-projective for each* $i \in I$*. The class of e-projective left* S*-semimodules is closed under direct sums.*

Proof. ($\Longrightarrow$) This implication follows by Lemma 2.11.

($\Longleftarrow$) Let $g : M \to N$ be a normal epimorphism and $f : \bigoplus_{i\in I} P_i \to N$ be an S-linear map. For every $j \in I$, there exists an S-linear map $h_j : P_j \to M$ such that $f \circ \iota_j = g \circ h_j$, where $\iota_j : P_j \longrightarrow \bigoplus_{i\in I} P_i$ is the canonical embedding.

$$\begin{tikzcd}
M \arrow[r, "g"] & N \arrow[r] & 0 \\
 & \bigoplus_{i\in I} P_i \arrow[u, "f"'] \arrow[ul, dashed, "h"'] & \\
 & P_j \arrow[u, "\iota_j"'] \arrow[uul, dashed, "h_j"] &
\end{tikzcd}$$

By the *Universal Property of Direct Coproducts*, there exists a unique S-linear map $h : \bigoplus_{i\in I} P_i \to M$ such that $h \circ \iota_j = h_j$ for every $j \in I$, i.e.

$$h : \bigoplus_{i\in I} P_i \to M, \ \sum_{i\in I} p_i \mapsto \sum_{i\in I} h_i(p_i).$$

Notice that h is S-linear and well defined since the sum $\sum_{i\in I} p_i$ is finite (all but finitely many of the coordinates are zero). Moreover, we have

$$\begin{aligned}(g \circ h)(\sum_{i \in I} p_i) &= g(\sum_{i \in I} h_i(p_i)) &&= \sum_{i \in I}(g \circ h_i)(p_i) \\ &= \sum_{i \in I}(f \circ \iota_i)(p_i) &&= f(\sum_{i \in I} \iota_i(p_i)) \\ &= f(\sum_{i \in I} p_i). \end{aligned}$$

Suppose that $h' : \bigoplus_{i \in I} P_i \to M$ is an S-linear map with $g \circ h' = f$. Then $f \circ \iota_j = g \circ h' \iota_j$ for every $j \in I$. Since P_j is e-projective for every $j \in I$, there exist S-linear maps $\tilde{h}_j, \hat{h}_j : P_j \to M$ such that $g \circ \tilde{h}_j = 0 = g \circ \hat{h}_j$ and $h_j + \tilde{h}_j = h'_j + \hat{h}_j$.

By the Universal Property of Direct Coproducts, there exist S-linear maps

$$\tilde{h}, \hat{h} : \bigoplus_{i \in I} P_i \to M, \ \tilde{h}(\sum_{i \in I} p_i) := \sum_{i \in I} \tilde{h}_i(p_i) \text{ and } \hat{h}(\sum_{i \in I} p_i) = \sum_{i \in I} \hat{h}_i(p_i).$$

$$\begin{tikzcd}
P \arrow[r, shift left, "\hat{h}"] \arrow[r, shift right, "\tilde{h}"'] & M \arrow[r, "g"] & N \arrow[r] & 0 \\
 & & \bigoplus_{i \in I} P_i \arrow[u, "f"'] \arrow[ul, "h"] \arrow[ul, bend right, "h'"'] & \\
 & & P_j \arrow[u, "\iota_j"'] \arrow[uul, bend left, "h_j"] &
\end{tikzcd}$$

Both maps are S-linear well defined since the sum $\sum_{i \in I} p_i$ is finite (all but finitely many of the coordinates are zero). Moreover, we have

$$\begin{aligned}(g \circ \tilde{h})(\sum_{i \in I} p_i) &= g(\sum_{i \in I} \tilde{h}_i(p_i)) = \sum_{i \in I}(g \circ \tilde{h}_i)(p_i) = 0; \\ \hat{h}(\sum_{i \in I} p_i) &= g(\sum_{i \in I} \hat{h}_i(p_i)) = \sum_{i \in I}(g \circ \hat{h}_i)(p_i) = 0 \end{aligned}$$

and

$$
\begin{aligned}
(h+\tilde{h})(\sum_{i\in I} p_i) &= h(\sum_{i\in I} p_i)+\tilde{h}(\sum_{i\in I} p_i) = \sum_{i\in I} h_i(p_i)+\sum_{i\in I}\tilde{h}_i(p_i)\\
&= \sum_{i\in I}(h_i+\tilde{h}_i)(p_i) = \sum_{i\in I}(h'_i+\hat{h}_i)(p_i)\\
&= (h'+\hat{h})(\sum_{i\in I} p_i).
\end{aligned}
$$

Hence $\bigoplus_{i\in I} P_i$ is M-e-projective. ■

Lemma 2.13 *Let P be a left S-semimodule. If*

$$0 \longrightarrow K \xrightarrow{\iota} L \xrightarrow{\pi} M \longrightarrow 0$$

is an exact sequence of left S-semimodules and P is L-e-projective, then P is K-e-projective and M-e-projective.

Proof. *Assume that P is L-e-projective.*

- ***Claim I:*** *P is M-e-projective.*

  Let $f : M \to N$ be a normal epimorphism and $g : P \to N$ an S-linear map.

$$
\begin{tikzcd}
L \arrow[r, "\pi"] & M \arrow[r, "f"] & N \arrow[r] & 0 \\
 & & P \arrow[u, "g"'] \arrow[ull, dashed, "h"] &
\end{tikzcd}
$$

  Since π and f are normal epimorphism, $f\circ\pi$ is a normal epimorphism as well (by Lemma 1.24 (2)(c)). Since P is L-e-projective, there exists an S-linear map $h : P \to M$ such that $f\circ\pi\circ h = g$. Then $\pi\circ h : P \to M$ is an S-linear map satisfying $f\circ(\pi\circ h) = g$.

Suppose that there exists an S-linear map $h' : P \to M$ such that $f \circ h' = g$. Since π is a normal epimorphism and P is L-e-projective, there exists an S-linear map $h^* : P \to L$ such that $\pi \circ h^* = h'$.

$$\begin{tikzcd}
L \arrow[r, "\pi"] & M \arrow[r] & 0 \\
& P \arrow[ul, dashed, "h^*"] \arrow[u, "h'"'] &
\end{tikzcd}$$

Moreover, $(f \circ \pi) \circ h^* = f \circ (\pi \circ h^*) = f \circ h' = g$. Since P is L-e-projective, there exist S-linear maps $h_1, h_2 : P \to L$ such that $f \circ \pi \circ h_1 = 0 = f \circ \pi \circ h_2$ and $h + h_1 = h^* + h_2$.

$$\begin{tikzcd}
P \arrow[r, shift left, "h_2"] \arrow[r, shift right, "h_1"'] & L \arrow[r, "\pi"] & M \arrow[r, "f"] & N \\
& & & P \arrow[ulll, "h^*"] \arrow[ulll, bend left, "h"] \arrow[ul, "h'"] \arrow[u, "g"']
\end{tikzcd}$$

Thus, $\pi \circ h_1$, $\pi \circ h_2 : P \to M$ are S-linear maps such that $f \circ \pi \circ h_1 = 0 = f \circ \pi \circ h_2$. Moreover,

$$\pi \circ h + \pi \circ h_1 = \pi \circ (h + h_1) = \pi \circ (h^* + h_2) = \pi \circ h^* + \pi \circ h_2 = h' + \pi \circ h_2.$$

Consequently, P is M-e-projective.

- ***Claim II:*** *P is K-e-projective.*

  Let $f : K \to N$ be a normal S-epimorphism and $g : P \to N$ an S-linear map. By Corollary 1.48, $(\iota', f'; Q := (N \oplus L)/\rho)$ is a pushout of (f, ι) such *that ρ is a congruence relation on $N \oplus L$ and*

$$\iota' : N \longrightarrow Q,\ n \mapsto [(n,0)]_\rho \text{ and } f' : L \longrightarrow Q,\ l \mapsto [(0,l)]_\rho.$$

$$\begin{tikzcd}
L \arrow[r, "f'"] & Q & \\
K \arrow[u, "\iota"] \arrow[r, "f"] & N \arrow[u, "\iota'"'] \arrow[r] & 0 \\
& P \arrow[u, "g"'] \arrow[uul, dashed, bend left, "h"] &
\end{tikzcd}$$

Since ι is a normal S-monomorphism and f is a normal S-epimorphism, it follows by Lemma 1.49 (2) & (4) that ι' is a normal monomorphism and by Lemma 1.49 (3) that f' is a normal epimorphism. Since f' is a normal epimorphism and P is L-e-projective, there exists and S-linear map $h : P \to L$ such that $f' \circ h = \iota' \circ g$.

Let $p \in P$. Since f is surjective, there exists $k \in K$ such that $f(k) = g(p)$. Notice that $(f' \circ \iota)(k) = (\iota' \circ f)(k) = (\iota' \circ g)(p) = (f' \circ h)(p)$. Since f' is k-normal, there exist $l_1, l_2 \in Ker(f')$ such that

$$\iota(k) + l_1 = h(p) + l_2. \tag{2.4}$$

Let

$$CP = (\iota^*, f^*; (N \oplus L)/\rho^*)$$

be the C-pushout of (f, ι) (defined in 1.46). Since Q is a pushout of (f, ι), there exists, by the Universal Property of Pushouts, an S-linear map φ :

$Q \to (N \oplus L)/\rho^*$ such that $\varphi \circ \iota' = \iota^*$ and $\varphi \circ f' = f^*$. Notice that for $i = 1, 2$:

$$[(0, l_i)]_{\rho^*} = f^*(l_i) = \varphi \circ f'(l_i) = \varphi(0) = [(0, 0)]_{\rho^*},$$

and so there exist $k_{i_1}, k_{i_2} \in K$ such that $f(k_{i_1}) = f(k_{i_2})$ and $l_i + \iota(k_{i_2}) = \iota(k_{i_1})$.

Since ι is a normal monomorphism, $\iota(K) \subseteq L$ is a subtractive subsemimodule, whence $l_1, l_2 \in \iota(K)$, i.e. $l_1 = \iota(k_1)$ and $l_2 = \iota(k_2)$ for some $k_1, k_2 \in K$. Rewriting (2.4) as $\iota(k) + \iota(k_1) = h(p) + \iota(k_2)$, we conclude that $h(p) \in \iota(K)$ (as ι is a normal monomorphism). Let $k_p \in K$ be such that $h(p) = \iota(k_p)$. Notice that this k_p is unique since ι is an injective. Therefore

$$h' : P \longrightarrow K, \ p \mapsto k_p$$

is well defined. Clearly, h' is S-linear. Now, for every $p \in P$, we have $(\iota' \circ f \circ h')(p) = (f' \circ (\iota \circ h'))(p) = (f' \circ \iota)(k_p) = (f' \circ h)(p) = (\iota' \circ g)(p)$, whence $(f \circ h')(p) = g(p)$ as ι' is injective.

Suppose that there exists an S-linear map $h^* : P \to K$ such that $f \circ h^* = g$. It follows that $f' \circ \iota \circ h^* = \iota' \circ f \circ h^* = \iota' \circ g$. Since P is L-e-projective, there exist S-linear maps $h_1, h_2 : P \to L$ such that $f' \circ h_1 = 0 = f' \circ h_2$ and $h + h_1 = \iota \circ h^* + h_2$. Let $p \in P$. For $i = 1, 2$, and since $h_i(p) \in Ker(f')$, there exists $k_p^i \in K$ such that $h_i(p) = \iota(k_p^i)$ (which is indeed unique as ι injective).

Then we have two well defined maps

$$h'_1 : P \longrightarrow K,\ p \mapsto k^1_p \text{ and } h'_2 : P \longrightarrow K,\ p \mapsto k^2_p.$$

which can be easily shown to be S-linear.

For every $p \in P$, and for $i = 1, 2$ we have $(\iota' \circ f \circ h'_i)(p) = (\iota' \circ f)(k^i_p) = (f' \circ \iota)(k^i_p) = (f' \circ h_i)(p) = 0$, whence $(f \circ h'_i)(p) = 0$ as ι' is injective. Moreover, we have

$$\begin{array}{rclcl}
\iota((h^* + h'_2)(p)) & = & (\iota \circ h^*)(p) + (\iota \circ h'_2)(p) & = & (\iota \circ h^*)(p) + \iota(k^2_p) \\
& = & (\iota \circ h^*)(p) + h_2(p) & = & (\iota \circ h^* + h_2)(p) \\
& = & (h + h_1)(p) & = & h(p) + h_1(p) \\
& = & \iota(k_p) + \iota(k^1_p) & = & (\iota \circ h')(p) + (\iota \circ h'_1)(p) \\
& = & \iota((h' + h'_1)(p)) & &
\end{array}$$

whence $(h^* + h'_2)(p) = (h' + h'_1)(p)$ as ι is injective. Consequently, P is K-e-projective.$\blacksquare$

2.14 *Let R be a ring, P a left R-module and $\{M_\lambda\}_{\lambda\in\Lambda}$ a collection of left S-semimodules such that P is M_λ-projective for every $\lambda \in \Lambda$. If $\Lambda = \{\lambda_1, \cdots, \lambda_k\}$ is finite, then P is $\bigoplus_{n=1}^{k} M_{\lambda_n}$-projective. If ${}_RP$ is finitely generated and Λ is arbitrary, then P is $\bigoplus_{\lambda\in\Lambda} M_\lambda$-projective (even if Λ is infinite).*

We provide a counter example showing that the corresponding result for semimodules does not hold for the notion of e-projectivity of semimodules over a semiring. The same example serves to show that the converse of Lemma 2.13 is not true (even when $M = L \oplus N$).

Example 2.15 *Let $S := M_2(\mathbb{R}^+) = E_1 \oplus E_2$, where*

$$E_1 = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\}, \ E_2 := \left\{ \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \mid c, d \in \mathbb{R}^+ \right\}$$

and consider

$$K = \left\{ \begin{bmatrix} u & u \\ v & v \end{bmatrix} \mid u, v \in \mathbb{R}^+ \right\} \text{ and } P := S/K.$$

Then

$$0 \to E_1 \xrightarrow{\iota_{E_1}} S \xrightarrow{\pi_{E_2}} E_2 \to 0$$

is exact, P is E_1-e-projective and E_2-e-projective. However, P is not S-e-projective (notice that $S = E_1 \oplus E_2$).

Proof. Since $E_1 \oplus E_2 = S$, it follows by the proof of Example 3.39 that P is not $(E_1 \oplus E_2)$-e-projective. Notice that E_1 and E_2 are ideal-simple left S-subsemimodules of S. Let $L \neq 0$ and $f : E_1 \to L$ be a normal S-epimorphism. Then $Ker(f) \subsetneqq E_1$, whence $Ker(f) = 0$ as E_1 is ideal-simple. Since f is k-normal and $Ker(f) = 0$, f is injective, whence an isomorphism. If $g : P \to L$ is an S-linear map, then $f^{-1} \circ g : P \to E_1$ is an S-linear map such that $f \circ f^{-1} \circ g = g$, and whenever there exists an S-linear map $h : P \to E_1$ such that $f \circ h = g$, we have $h = f^{-1} \circ f \circ h = f^{-1} \circ g$. Hence, P is E_1-e-projective. Similarly, one can prove that P is E_2-e-projective.$\blacksquare$

2.2 Injective Semimodules

There are several notions of injectivity for a semimodules over a semiring which coincide if it were a module over a ring. In this Chapter, we consider some of these and clarify the relationships between them. In particular, we investigate the so called *e-injective semimodules* which turn to coincide with the so called *normally injective semimodules* (both notions introduced by Abuhlail [3, 1.25, 1.24] and called *uniformly injective semimodules*; the terminology "*e-injective*" was first used in [6]).

As before, $(S, +, 0, \cdot, 1)$ is a semiring and, unless otherwise explicitly mentioned, an S-module is a **left** S-semimodule. Exact sequences here are in the sense of Abuhlail [1] (see Definition 1.12).

Definition 2.2 *([3, 1.24]) A left S-semimodules J is M-e-**injective** (where M is a left S-semimodule) if the contravariant functor*

$$Hom_S(-, J) : \ {}_S\mathbf{SM} \longrightarrow \ {}_{\mathbb{Z}^+}\mathbf{SM}$$

transfers every short exact sequence of left S-semimodules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

into a short exact sequence of commutative monoids

$$0 \longrightarrow Hom_S(N,J) \longrightarrow Hom_S(M,J) \longrightarrow Hom_S(L,J) \longrightarrow 0.$$

We say that J is e***-injective*** *if J is M-e-injective for every left S-semimodule M.*

2.16 *Let I be a left S-semimodule.*

For a left S-semimodule M, we say that I is

*M**-injective*** *[18, page 197] if for every* injective *S-linear map $f : L \to M$ and any S-linear map $g : L \to I$, there exists an S-linear map $h : M \to I$ such that $h \circ f = g$;*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[dl, dotted, "h"] \\ & I & \end{tikzcd}$$

*M-i**-injective*** *[12, Definition 6] if for every* normal monomorphism *$f : L \to M$ and any S-linear map $g : L \to I$, there exists an S-linear map $h : M \to I$ such that $h \circ f = g$;*

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[rr, "f(normal)"] \arrow[d, "g"'] & & M \arrow[dll, dotted, "h"] \\ & I & & \end{tikzcd}$$

normally *M**-injective*** *[3, 1.24] if for every* normal monomorphism *$f : L \longrightarrow M$ and any S-linear map $g : L \longrightarrow I$, there exists an S-linear map*

$h : M \longrightarrow I$ such that $h \circ f = g$

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f(normal)"] \arrow[d, "g"'] & M \arrow[r, shift left, "h_2"] \arrow[r, shift right, "h_1"'] \arrow[dl, bend right, "h'"'] \arrow[dl, dotted, "h"] & I \\
& I & &
\end{tikzcd}$$

and whenever an S-linear map $h' : M \to I$ satisfies $h' \circ f = g$, there exist S-linear maps $h_1, h_2 : M \to I$ such that $h_1 \circ f = 0 = h_2 \circ f$ and $h + h_1 = h' + h_2$.

We say that I is injective (resp., i-injective, normally injective) if I is M-injective (resp., M-i-injective, normally M-projective) for every left S-semimodule M.

Proposition 2.17 *Let I be a left S-semimodule.*

(1) *I is M-e-injective (where M is a left S-semimodule) if and only if I is normally M-injective.*

(2) *${}_S I$ is e-injective if and only if ${}_S I$ is normally injective.*

Proof. We only need to prove (1). Let M be a left S-semimodule.

($\Longrightarrow$) Assume that I is M-e-injective. Let $L \leq_S M$ be a subtractive S-subsemimodule. By Lemma 1.29, we have a short exact sequence of left S-semimodules

$$0 \longrightarrow L \stackrel{\iota}{\longrightarrow} M \stackrel{\pi}{\longrightarrow} M/L \longrightarrow 0 \tag{2.5}$$

where ι is the canonical embedding and π is the canonical projection. By our assumption, the contravariant functor $Hom_S(-, J) : {}_S\mathbf{SM} \longrightarrow {}_{\mathbb{Z}^+}\mathbf{SM}$ preserves exact sequences, whence the following sequence of commutative monoids

$$0 \longrightarrow Hom_S(M/L, J) \overset{(\pi,J)}{\longrightarrow} Hom_S(M, J) \overset{(\iota,J)}{\longrightarrow} Hom_S(L, J) \longrightarrow 0$$

is exact. In particular, $(\iota, J) : Hom_S(M, J) \longrightarrow Hom_S(L, J)$ is a normal epimorphism, *i.e.* J is normally M-injective.

($\Longleftarrow$) Let

$$0 \longrightarrow L \overset{f}{\longrightarrow} M \overset{g}{\longrightarrow} N \longrightarrow 0 \tag{2.6}$$

be an exact sequence of left S-semimodules. Applying the contravariant functor $Hom_S(-, J)$ to (2.6) it follows by Lemma 1.42 (2) and our assumption that the following sequence of commutative monoids

$$0 \longrightarrow Hom_S(N, J) \overset{(g,J)}{\longrightarrow} Hom_S(M, J) \overset{(f,J)}{\longrightarrow} Hom_S(L, J) \longrightarrow 0$$

is exact, i.e. ${}_SJ$ is exact.■

Remark 2.18 *It is obvious from the definitions that injective and e-injective semimodules are i-injective.*

While every projective semimodule is e-projective as shown in [9, Theorem 4], it is not evident that every injective semimodule is e-injective if the base semiring is arbitrary. However, we have a partial result:

Proposition 2.19 ([6, Theorem 4.5]) *Let S be an additively idempotent semiring.*

(1) *Every left S-semimodule can be embedded in an e-injective left semimodule;*

(2) *Every injective left S-semimodule is e-injective.*

There are plenty of examples of injective semimodules which are not e-injective.

Example 2.20 *([6, 4.6]) Let D be an additively idempotent division semiring (e.g., $\mathbb{B}$). Then D has an e-injective left S-semimodule which is not injective.*

Lemma 2.21 (cf. [6, Corollary 3.3])

(1) *Let M be a left S-semimodule. Every retract of a left M-e-injective S-semimodule is M-e-injective.*

(2) *A retract of an e-injective S-semimodule is e-injective.*

Proof. *We need to prove (1) only.*

Let J be an M-e-injective left S-semimodule and I a retract of J along with S-linear maps $\iota : I \longrightarrow J$ and $\pi : J \longrightarrow I$ such that $\pi \circ \iota = id_I$. Let $f : L \to M$ be a normal S-monomorphism and $g : L \to I$ be an S-linear map.

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[ddl, dashed, "h^*"] \\ & I \arrow[d, "\iota"'] & \\ & J & \end{tikzcd}$$

Since J is M-e-injective, there is an S-linear map $h^* : M \to J$ such that $h^* \circ f = \iota \circ g$. Consider $h := \pi \circ h^*$. Then we have $h \circ f = (\pi \circ h^*) \circ f = \pi \circ (h^* \circ f) = \pi \circ (\iota \circ g) = (\pi \circ \iota) \circ g = id_I \circ g = g$.

Suppose that $h' : M \to I$ is an S-linear map such that $h' \circ f = g$. Notice that $\iota \circ h' \circ f = \iota \circ g$. Since J is M-e-injective, there exist S-linear maps $h_1^*, h_2^* : M \to J$ such that $h_1^* \circ f = 0 = h_2^* \circ f$ and $h^* + h_1^* = \iota \circ h' + h_2^*$.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[r, shift left, "h_2^*"] \arrow[r, shift right, "h_1^*"'] \arrow[dl, bend right, "h'"'] \arrow[dl, "h"] \arrow[ddl, bend left, "h^*"] & J \\
& I \arrow[d, "\iota"'] & & \\
& J & &
\end{tikzcd}$$

Consider $h_1 := \pi \circ h_1^*$ and $h_2 := \pi \circ h_2^*$. Then we have, for $i = 1, 2$, $h_i \circ f = \pi \circ h_i^* \circ f = \pi \circ 0 = 0$. Moreover, we have $h + h_1 = \pi \circ \iota \circ h + \pi \circ h_1^* = \pi \circ (\iota \circ h + h_1^*) = \pi \circ (\iota \circ h' + h_2^*) = \pi \circ \iota \circ h' + \pi \circ h_2^* = h' + h_2$.

Proposition 2.22 (cf. [6, Corollary 3.3]) *Let M be a left S-semimodule and $\{J_\lambda\}_{\lambda \in \Lambda}$ be a collection of left S-semimodules. Then $\prod_{\lambda \in \Lambda} J_\lambda$ is (M)-e-injective if and only if J_λ is M-e-injective for every $\lambda \in \Lambda$.*

Proof. Let $J := \prod_{\lambda \in \Lambda} J_\lambda$ and, for each $\lambda \in \Lambda$, let $\iota_\lambda : J_\lambda \longrightarrow J$ and $\pi_\lambda : J \longrightarrow J_\lambda$ be the canonical S-linear maps.

($\Longrightarrow$) For each $\lambda \in \Lambda$, we have $\pi_\lambda \circ \iota_\lambda = id_{J_\lambda}$, *i.e.* J_λ is a retract of J. The result follows from Lemma 2.21.

($\Longleftarrow$) Assume that J_λ is M-e-injective for every $\lambda \in \Lambda$. Let $f : L \to M$ be a normal monomorphism and $g : L \to J$ an S-linear map.

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[ddl, dashed, "h_\lambda^*"] \\ & J \arrow[d, "\pi_\lambda"'] & \\ & J_\lambda & \end{tikzcd}$$

Since J_λ is M-e-injective for each $\lambda \in \Lambda$, there is an S-linear map $h_\lambda^* : M \to J_\lambda$ such that $h_\lambda^* \circ f = \pi_\lambda \circ g$. By the *Universal Property of Direct Products*, there exists an S-linear map

$$h : M \longrightarrow J,\ m \mapsto \prod_{\lambda \in \Lambda} (\iota_\lambda \circ h_\lambda^*)(m).$$

Notice that for every $l \in L$, we have

$$(h \circ f)(l) = \prod_{\lambda \in \Lambda} (\iota_\lambda \circ h_\lambda^*)(f(l)) = \prod_{\lambda \in \Lambda} (\iota_\lambda \circ \pi_\lambda)(g(l)) = g(l).$$

Suppose that there exists an S-linear map $h' : M \to J$ such that $h' \circ f = g$. It follows that $\pi_\lambda \circ h' \circ f = \pi_\lambda \circ g$ for every $\lambda \in \Lambda$. Since J_λ is M-e-injective, there exist S-linear maps $h_{1_\lambda}^*, h_{2_\lambda}^* : M \to J$ such that $h_{1_\lambda}^* \circ f = 0 = h_{2_\lambda}^* \circ f$ and

$h^*_\lambda + h^*_{1_\lambda} = \pi_\lambda \circ h' + h^*_{2_\lambda}$.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "g"'] & M \arrow[r, shift left, "h^*_{1_\lambda}"] \arrow[r, shift right, "h^*_{2_\lambda}"'] \arrow[dl, bend right, "h'"'] \arrow[dl, "h"] \arrow[ddl, bend left, "h^*_\lambda"] & J_\lambda \\
& J \arrow[d, "\pi_\lambda"'] & & \\
& J_\lambda & &
\end{tikzcd}$$

For $i = 1, 2$, there exists by the *Universal Property of Direct Products* an S-linear map

$$h_i : M \longrightarrow J,\ m \mapsto \prod_{\lambda\in\Lambda}(\iota_\lambda \circ h^*_{i_\lambda})(m).$$

For $i = 1, 2$ and every $l \in L$ we have $(h_i \circ f)(l) = \prod_{\lambda\in\Lambda}(\iota_\lambda \circ h^*_{i_\lambda})(f(l)) = \prod_{\lambda\in\Lambda}\iota_\lambda(0) = 0$. Moreover, we have for every $m \in M$:

$$\begin{aligned}
(h + h_1)(m) &= \prod_{\lambda\in\Lambda}(\iota_\lambda \circ \pi_\lambda \circ h)(m) + \prod_{\lambda\in\Lambda}(\iota_\lambda \circ h^*_{1_\lambda})(m) \\
&= \prod_{\lambda\in\Lambda}(\iota_\lambda \circ (\pi_\lambda \circ h + h^*_{1_\lambda}))(m) \\
&= \prod_{\lambda\in\Lambda}(\iota_\lambda \circ (h^*_\lambda + h^*_{1_\lambda}))(m) \\
&= \prod_{\lambda\in\Lambda}(\iota_\lambda \circ (\pi_\lambda \circ h'^*_{2_\lambda}))(m) \\
&= \prod_{\lambda\in\Lambda}(h' + \iota_\lambda \circ h^*_{2_\lambda})(m) \\
&= (h' + h_2)(m). \ \blacksquare
\end{aligned}$$

Lemma 2.23 *Let*

$$0 \longrightarrow L \xrightarrow{p} M \xrightarrow{q} N \to 0$$

be a short exact sequence of left S-semimodules. If a left S-semimodule J is M-e-injective, then J is L-e-injective and N-e-injective.

Proof. Let J be a left S-semimodule.

Claim I: J is L-e-injective.

Let $f : K \to L$ be a normal monomorphism and $g : K \to J$ an S-linear map. Clearly, $p \circ f$ is a normal monomorphism.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "g"'] & L \arrow[r, "p"] & M \arrow[dll, dashed, "h^*"] \\ & J & & \end{tikzcd}$$

Since J is M-e-injective, there exists an S-linear map $h^* : M \to J$ such that $h^* \circ p \circ f = g$. Consider $h := h^* \circ p : L \longrightarrow J$. Then $h \circ f = h^* \circ p \circ f = g$.

Suppose now that $h' : L \to J$ is an S-linear map such that $h' \circ f = g$. Since $p : L \longrightarrow M$ is a normal monomorphism and J is M-e-injective, there exists an S-linear map $\tilde{h} : M \to J$ such that $\tilde{h} \circ p = h'$. Since $\tilde{h} \circ p = h^* \circ p = g$, there exist S-linear maps $h_1^*, h_2^* : M \to J$ such that $h_1^* \circ p = 0 = h_2^* \circ p$ and $h^* + h_1^* = \tilde{h} + h_2^*$. Considering $h_1 := h_1^* \circ p$ and $h_2 = h_2^* \circ p$, we have $h_1 \circ f = h_1^* \circ p \circ f = 0 = h_2^* \circ p \circ f = h_2 \circ f$ and $h + h_1 = h^* \circ p + h_1^* \circ p = (h^* + h_1^*) \circ p = (\tilde{h} + h_2^*) \circ p = \tilde{h} \circ p + h_2^* \circ p = h' + h_2$.

Claim II: Let $f : K \to N$ be a normal monomorphism and $g : K \to J$ an

S-linear map.

$$\begin{tikzcd}
 & U \arrow[r, "f'"] \arrow[d, "q'"'] & M \arrow[d, "q"] \arrow[ddl, dashed, bend left, "h^*"] \\
0 \arrow[r] & K \arrow[r, "f"] \arrow[d, "g"'] & N \\
 & J &
\end{tikzcd}$$

Let $(U; q', f')$ be a pullback of (q, f) (see 1.45). Clearly, f' is a normal S-monomorphism. Since J is M-e-injective, there exists an S-linear map $h^* : M \to J$ such that $h^* \circ f' = g \circ q'$. Let $n \in N$. Since q is surjective, there exists $m_n \in M$ such that $n = q(m_n)$. Define

$$h : N \to J,\ n \mapsto h^*(m_n).$$

We claim that h is well defined. Suppose that there exists another $m \in M$ such that $q(m) = n = q(m_n)$. Since q is k-normal, there exist $m_1, m_2 \in Ker(q)$ such that $m + m_1 = m_n + m_2$. Since $m_1, m_2 \in Ker(q)$, $(m_1, 0), (m_2, 0) \in U$ and so for $i = 1, 2$ we have $h^*(m_i) = (h^* \circ f')(m_i, 0) = (g \circ q')(m_i, 0) = g(0) = 0$, where $h^*(m) = h^*(m_n)$. Thus h well defined as a map. Clearly, h is S-linear. Moreover, for every $k \in K$ we have $f(k) = q(m_{f(k)})$ for some $m_{f(k)} \in M$, thus $(m_{f(k)}, k) \in U$ and it follows that

$$\begin{aligned}
(h \circ f)(k) &= (h \circ f \circ q')(m_{f(k)}, k) &&= (h \circ q \circ f')(m_{f(k)}, k) \\
&= (h^* \circ f')(m_{f(k)}, k) &&= (g \circ q')(m_{f(k)}, k) \\
&= g(k),
\end{aligned}$$

i.e. $h \circ f = g$.

Suppose that there exists an S-linear map $h' : N \to J$ such that $h' \circ f = g$. Notice that $h' \circ q \circ f' = h' \circ f \circ q' = g \circ q'$. Since J is M-e-injective, there exist $h_1^*, h_2^* : M \to J$ such that $h_1^* \circ f' = 0 = h_2^* \circ f'$ and $h^* + h_1^* = h' \circ q + h_2^*$.

Let $n \in N$. Since q is surjective, there exists $m_n \in M$ such that $q(m_n) = n$. Define

$$h_1 : N \to J,\ n \mapsto h_1^*(m_n) \text{ and } h_2 : N \to J,\ n \mapsto h_2^*(m_n).$$

One can prove as above that h_1 and h_2 are well-defined. It is clear that both h_1 and h_2 are S-linear. Notice that for every $k \in K$, we have $(m_{f(k)}, k) \in U$ whence, for $i = 1, 2$, $(h_i \circ f)(k) = (h_i \circ f \circ q)(m_{f(k)}, k) = (h_i \circ q \circ f')(m_{f(k)}, k) = (h_i^* \circ f')(m_{f(k)}, k) = 0$. Moreover, for every $n \in N$, we have

$$\begin{array}{rcccl} (h + h_1)(n) & = & h(n) + h_1(n) & = & h^*(m_n) + h_1^*(m_n) \\ & = & (h^* + h_1^*)(m_n) & = & (h' \circ q + h_2^*)(m_n) \\ & = & (h' \circ q)(m_n) + h_2^*(m_n) & = & h'(n) + h_2(n) \\ & = & (h' + h_2)(n). \blacksquare & & \end{array}$$

Remark 2.24 *The converse of Lemma 2.23 is not true in general as will be shown in Example 2.28.*

2.2.1 Example

This subsection is devoted to an example of a semiring S over which the class of S-i-injective left S-semimodules is *strictly larger* than that of S-e-injective left S-semimodules.

Throughout this subsection,

$$S := M_2(\mathbb{R}^+).$$

Lemma 2.25 *The only non-trivial subtractive left ideals of S are*

$$\begin{aligned}
E_1 &= Span\left(\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}\right) = \left\{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{R}^+\right\} \\
E_2 &= Span\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R}^+\right\} \\
N_r &= \left\{\begin{bmatrix} ra & a \\ rb & b \end{bmatrix} \mid a, b \in \mathbb{R}^+\right\}, \ r \in \mathbb{R}^+ \backslash \{0\}.
\end{aligned}$$

Proof. E_1 is a left ideal since for every $a, b, c, d, p, q, r, s \in \mathbb{R}^+$ we have

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} + \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} pa + qb + c & 0 \\ ra + sb + d & 0 \end{bmatrix} \in E_1.$$

Moreover, E_1 is subtractive since

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} + \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}$$

implies $q = 0 = s$ and $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in E_1$. Similarly, we have E_2 is subtractive.

For any nonzero $r \in \mathbb{R}^+$, N_r is a left ideal since

$$\begin{bmatrix} k & l \\ m & n \end{bmatrix} \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} + \begin{bmatrix} rc & c \\ rd & d \end{bmatrix} = \begin{bmatrix} r(ka+lb+c) & ka+lb+c \\ r(ma+nb+d) & ma+nb+d \end{bmatrix} \in N_r$$

for all $a, b, c, d, k, l, m, n \in \mathbb{R}^+$. It is subtractive since

$$\begin{bmatrix} k & l \\ m & n \end{bmatrix} + \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} = \begin{bmatrix} rc & c \\ rd & d \end{bmatrix}$$

implies $c = a + k/r = a + l, d = b + m/r = b + n$, which implies $k = rl, m = rn$, and $\begin{bmatrix} k & l \\ m & n \end{bmatrix} \in N_r$.

If I is a subtractive left ideal of $M_2(\mathbb{R}^+)$ such that $E_1 \subsetneqq I$, then there exists $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in I$ such that $q \neq 0$ or $s \neq 0$, which implies $\begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} \in I$ as $\begin{bmatrix} p & 0 \\ r & 0 \end{bmatrix} \in I$ and

$$\begin{bmatrix} p & 0 \\ r & 0 \end{bmatrix} + \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in I$$

If $q \neq 0$, then

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1/q & 0 \end{bmatrix} \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix}.$$

If $s \neq 0$, then

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1/s \end{bmatrix} \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix}.$$

Either way $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in I$, which implies $E_1 \subseteq I$ and $I = S$.

Similarly, if I is a subtractive left ideal of $M_2(\mathbb{R}^+)$ such that $E_2 \subsetneqq I$, then $I = S$.

Let $r \in \mathbb{R}^+ \backslash \{0\}$ and I be a subtractive left ideal of $M_2(\mathbb{R}^+)$ such that $N_r \subsetneqq I$. Then there exists $\begin{bmatrix} k & l \\ m & n \end{bmatrix} \in I$ such that $k \neq rl$ or $m \neq rn$. Without loss of generality, let $k < rl$. Then $k + p = rl$ for some $p \in \mathbb{R}^+ \backslash \{0\}$. Thus $\begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix} \in I$ or $\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \in I$ for some $q \in \mathbb{R}^+$ as

$$\begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix} + \begin{bmatrix} k & l \\ m & n \end{bmatrix} = \begin{bmatrix} rl & l \\ rn & n \end{bmatrix} \in I$$

or

$$\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} + \begin{bmatrix} k & l \\ m & n \end{bmatrix} = \begin{bmatrix} rl & l \\ m & m/r \end{bmatrix} \in I.$$

Thus

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}.$$

Either way we have $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in I$, which implies $E_1 \subsetneqq I$ and $I = S$.

Let I be a non-zero subtractive left ideal of $M_2(\mathbb{R}^+)$. Then $\begin{bmatrix} k & l \\ m & n \end{bmatrix} \in I \backslash \{0\}$ for some $k, l, m, n \in \mathbb{R}^+$. If $k \neq 0$, then

$$\begin{bmatrix} 1/k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k & l \\ m & n \end{bmatrix} = \begin{bmatrix} 1 & l/k \\ 0 & 0 \end{bmatrix}$$

whence $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in I$, or $\begin{bmatrix} k/l & 1 \\ 0 & 0 \end{bmatrix} \in I$; and so, $I \in \{E_1, N_{k/l}, S\}$ as I contains E_1 or $N_{k/l}$. If $l \neq 0$, then

$$\begin{bmatrix} 0 & 0 \\ 1/l & 0 \end{bmatrix} \begin{bmatrix} k & l \\ m & n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ k/l & 1 \end{bmatrix},$$

whence $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in I$ or $\begin{bmatrix} 0 & 0 \\ k/l & l \end{bmatrix} \in I$; and so $I \in \{E_2, N_{k/l}, S\}$ as I contains

E_2 or $N_{k/l}$. If $m \neq 0$, then

$$\begin{bmatrix} 0 & 1/m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k & l \\ m & n \end{bmatrix} = \begin{bmatrix} 1 & n/m \\ 0 & 0 \end{bmatrix}$$

whence $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in I$, or $\begin{bmatrix} m/n & 1 \\ 0 & 0 \end{bmatrix} \in I$; and so $I \in \{E_1, N_{m/n}, S\}$ as I contains E_1 or $N_{m/n}$. If $n \neq 0$, then

$$\begin{bmatrix} 0 & 0 \\ 0 & 1/n \end{bmatrix} \begin{bmatrix} k & l \\ m & n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ m/n & 1 \end{bmatrix}$$

whence $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in I$, or $\begin{bmatrix} 0 & 0 \\ m/n & l \end{bmatrix} \in I$; and so, $I \in \{E_2, N_{m/n}, S\}$ as I contains E_2 or $N_{m/n}$.■

Lemma 2.26 *Every left S-semimodule is S-i-injective.*

Proof. Let M be a left S-semimodule, $f : N \to S$ be a normal S-monomorphism, and $g : N \to M$ be an S-linear map. Then $f(N)$ is subtractive, whence $f(N) \in \{0, E_1, E_2, S\}$ or $f(N) = N_r$ for some $r \in \mathbb{R}^+ \backslash \{0\}$. If $f(N) = 0$, then choose $0 = h : S \to M$, thus $g = h \circ f$. If $f(N) = S$, then f is an S-isomorphism and choose $h = g \circ f^{-1}$, thus $g = h \circ f$. If $f(N) = E_1$, then there exists a unique $n_0 \in N$ such that

$$f(n_0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Choose $h : S \to M$ with

$$h(\begin{bmatrix} p & q \\ r & s \end{bmatrix}) = \begin{bmatrix} p & q \\ r & s \end{bmatrix} g(n_0).$$

Let $n \in N$. Then

$$f(n) = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} f(n_0) = f(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} n_0)$$

for some $a, b \in \mathbb{R}^+$. Since f is injective, $n = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} n_0$. It follows that

$$(h \circ f)(n) = h(f(n)) = h(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}) = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} g(n_0) = g(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} n_0) = g(n).$$

Similarly, if $f(N) = E_2$, then $h \circ f = g$.

If $f(N) = N_r$ for some $r \in \mathbb{R}^+ \backslash \{0\}$, then there exists a unique $n_0 \in N$ such that

$$f(n_0) = \begin{bmatrix} 1 & 1/r \\ 0 & 0 \end{bmatrix}.$$

Choose $h : S \to M$ with

$$h(\begin{bmatrix} j & k \\ l & m \end{bmatrix}) = \begin{bmatrix} j & k \\ l & m \end{bmatrix} g(n_0).$$

For every $n \in N$, we have

$$f(n) = \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} = \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} f(n_0) = f(\begin{bmatrix} ra & a \\ rb & b \end{bmatrix} n_0)$$

for some $a, b \in \mathbb{R}^+$. Since f is 1-1, $n = \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} n_0$ and so

$$\begin{aligned} (h \circ f)(n) &= h(f(n)) = h\left(\begin{bmatrix} ra & a \\ rb & b \end{bmatrix}\right) \\ &= \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} g(n_0) = g\left(\begin{bmatrix} ra & a \\ rb & b \end{bmatrix} n_0\right) \\ &= g(n). \blacksquare \end{aligned}$$

We are now ready to provide an example of an S-i-injective semimodule which is not S-e-injective.

Example 2.27 *The left S-semimodule*

$$N_1 = \left\{ \begin{bmatrix} a & a \\ b & b \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\} \tag{2.7}$$

is S-i-injective but not S-e-injective.

Proof. Let $\iota : N_1 \to S$ be an embedding and $id : N_1 \to N_1$ be the identity map.

Since N_1 is subtractive, ι is a normal S-monomorphism. Let $h_1, h_2 : S \to N_1$ with

$$h_1\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) = \begin{bmatrix} p & p \\ r & r \end{bmatrix} \text{ and } h_2\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) = \begin{bmatrix} q & q \\ s & s \end{bmatrix}.$$

Then

$$(h_1 \circ \iota)\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right) = h_1\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right) = \begin{bmatrix} a & a \\ b & b \end{bmatrix} = id\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right)$$

and

$$(h_2 \circ \iota)\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right) = h_2\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right) = \begin{bmatrix} a & a \\ b & b \end{bmatrix} = id\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right).$$

Suppose that there exist $k_1, k_2 : S \to N_1$ such that $k_1 \circ \iota = 0 = k_2 \circ \iota$ and $h_1 + k_1 = h_2 + k_2$. Write

$$k_1\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} l & m \\ n & o \end{bmatrix} \text{ and } k_2\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

for some $k, l, m, n, o, p, q, r, s \in \mathbb{R}^+$. Then

$$k_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} l & m \\ n & o \end{bmatrix} \text{ and } k_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

for every $a, b, c, d \in \mathbb{R}^+$. It follows that

$$0 = (k_1 \circ \iota)\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} l & m \\ n & o \end{bmatrix} = \begin{bmatrix} l+n & m+o \\ 0 & 0 \end{bmatrix},$$

which implies that $l = m = n = o = 0$ as 0 is the only element of $\mathbb{R}^+$ which has additive inverse. So,

$$\begin{aligned} 0 &= (k_2 \circ \iota)\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) &= k_2\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} &= \begin{bmatrix} p+r & q+s \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

which implies that $p = q = r = s = 0$ as 0 is the only element of $\mathbb{R}^+$ which has additive inverse. Thus $k_1 = 0 = k_2$, contradiction with $h_1 + k_1 = h_2 + k_2$ as $h_1 \neq h_2$. Hence, N_1 is not S-e-injective.■

The following example shows that the converse of Lemma 2.23 is not true in general.

Example 2.28 *Consider the short exact sequence*

$$0 \to E_1 \xrightarrow{\iota_{E_1}} S \xrightarrow{\pi_{E_2}} E_2 \to 0$$

of left S-semimodules. Then N_1 is E_1-e-injective and E_2-e-injective but not S-e-injective.

Proof. Let $f : M \to E_1, g : M \to N_1$ be S-linear maps where f is a normal monomorphism. If f is the zero map, then we are done. If f is not zero, then f is an isomorphism as E_1 is ideal-simple. Define $h = g \circ f^{-1}$. Then $h \circ f = (g \circ f^{-1}) \circ f = g$. If $h' : E_1 \to N_1$ is an S-linear map satisfies $h' \circ f = g$, then $h' = h' \circ (f \circ f^{-1}) = g \circ f^{-1} = h$. Hence N_1 is E_1-e-injective. Similarly, N_1 is E_2-e-injective. However, N_1 is not S-e-injective as shown in Example 2.27.

2.2.2 The Embedding Problem

It is well-known that the category of modules over a ring has enough injectives, *i.e.* every module has an injective hull. This is true for the categories of semimodules over some semirings, e.g., the *additively idempotent semirings* [18, Corollary 17.34]. In fact, the situation over some semirings can be extremely bad:

Lemma 2.29 *If S is an entire, cancellative, zerosumfree semiring, then the only injective left S-semimodule is* $\{0\}$ ([18, Proposition 17.21]).

Example 2.30 *The category of commutative monoids (i.e., $\mathbb{Z}^+$-semimodules) has no non-zero injective objects.*

Another significant difference is that Baer's Criterion (a left module M over a ring R is injective if M is R-injective) is not valid for semimodules over arbitrary semirings (which are not rings).

Lemma 2.31 ([24, Theorem 3]) *If S satisfies the Baer's criterion and every left S-semimodule is embedded in an injective left S-semimodule, then S is a ring.*

2.32 *We define a left S-semimodule N to be* ***divisible*** *if for every $s \in S$, which is not a zero divisor, there exists for every $n \in N$ some $m_n \in N$ such that $sm_n = n$. As in the case of modules over a ring, every injective semimodule over a semiring is divisible.*

The proof of the following observation is similar to that in the case of modules over rings [40, 16.6].

Lemma 2.33 *Every S-injective left S-semimodule is divisible.*

Proof. Let N be an injective left S-semimodule and $n \in N$. Let $s \in S$ be a non zero-divisor. Claim: there exists $m_n \in N$ such that $sm_n = n$. Consider the canonical embedding $0 \longrightarrow Ss \xrightarrow{\iota} S$ and the S-linear map

$$h : Ss \longrightarrow N,\ ts \mapsto tn.$$

By our assumption, N is S-injective, whence there exists an S-linear map $g : S \longrightarrow N$ such that $g \circ \iota = h$. Let $m_n := g(1_S)$. Then we have

$$n = h(s) = (g \circ \iota)(s) = g(s) = g(s \cdot 1_S) = sg(1_S) = sm_n. \blacksquare$$

The converse of Lemma 2.33 is not true in general as the following example shows.

Example 2.34 $\mathbb{Q}$ *is a divisible commutative monoid which is not injective.*

2.35 *Let R be a ring. Every left R-module can be embedded in an injective module $Hom_{\mathbb{Z}}(R, D)$, (see [19, page 407, 421]). For a semiring S, we prove that every left S-semimodule can be embedded into $Hom_{\mathbb{Z}^+}(S, D)$ for some divisible commutative monoid D. However, it is unknown whether $Hom_{\mathbb{Z}^+}(S, D)$ is necessarily e-injective.*

Lemma 2.36 *Every commutative monoid can be embedded subtractively into a divisible commutative monoid.*

Proof. Let B be a commutative monoid. Then there exists a surjective morphism of monoids $f : \mathbb{Z}^{+(\Lambda)} \to B$ for some index set Λ. Let g be the embedding of $\mathbb{Z}^{+(\Lambda)}$ into $\mathbb{Q}^{+(\Lambda)}$. Let $(g', f'; P)$ be a pushout of (f, g).

$$\begin{array}{ccc} \mathbb{Q}^{+(\Lambda)} & \xrightarrow{f'} & P \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g'} \\ \mathbb{Z}^{+(\Lambda)} & \xrightarrow[f]{} & B \end{array}$$

Notice that g' is subtractive since g is subtractive. Moreover, the commutative monoid P is divisible since for every $n \in \mathbb{Z}^+$ and $p \in P$ we have $(q_\lambda)_\Lambda, (q'_\lambda)_\Lambda \in \mathbb{Q}^+$ such that $p = f'((q_\lambda)_\Lambda)$ and $nq'_\lambda = q_\lambda$. Thus $nf'((q'_\lambda)_\Lambda) = f'((nq'_\lambda)_\Lambda) = f'((q_\lambda)_\Lambda)$.

Let $C := \{q \in \mathbb{Q}^+ | 0 \leq q < 1\}$. Then $B \oplus C^{(\Lambda)}$ is a commutative monoid with

$$(b,(c_\lambda)) + (b',(c'_\lambda)) = (b + b' + f((\lfloor c_\lambda + c'_\lambda \rfloor)_\Lambda), (c_\lambda + c'_\lambda - \lfloor c_\lambda + c'_\lambda \rfloor)).$$

$$\begin{tikzcd}
 & & B \oplus C^{(\Lambda)} \\
\mathbb{Q}^{+(\Lambda)} \arrow[r, "f'"] \arrow[urr, bend left, "f^*"] & P \arrow[ur, dashed, "\varphi"] & \\
\mathbb{Z}^{+(\Lambda)} \arrow[u, "g"] \arrow[r, "f"'] & B \arrow[u, "g'"'] \arrow[uur, bend right, "g^*"'] &
\end{tikzcd}$$

The map

$$g^* : B \longrightarrow B \oplus C^{(\Lambda)},\ b \mapsto (b, 0)$$

is a $\mathbb{Z}^+$-monomorphism. The map

$$f^* : \mathbb{Q}^{+(\Lambda)} \longrightarrow B \oplus C^{(\Lambda)},\ (q_\lambda) \mapsto (f((\lfloor q_\lambda \rfloor)_\Lambda), (q_\lambda - \lfloor q_\lambda \rfloor)_\Lambda)$$

is a $\mathbb{Z}^+$-homomorphism. Since $f^* \circ g = g^* \circ f$, there exists, by the *Universal Property of Pushouts*, a unique map $\varphi : P \to B \oplus C^{(\Lambda)}$ such that $\varphi \circ f' = f^*$ and $\varphi \circ g' = g^*$. Since g^* is injective, g' is injective. Hence $g' : B \longrightarrow P$ is a normal $\mathbb{Z}^+$-monomorphism from B into the divisible commutative monoid P. ■

Lemma 2.37 *Every left S-semimodule can be embedded into $Hom_{\mathbb{Z}^+}(S, D)$ for some divisible commutative monoid D.*

Proof. Let M be a left S-semimodule. By Lemma 2.36 there exists a normal monomorphism of commutative monoids $\mu : M \to D$ for some divisible commutative monoid D. Consider the canonical S-linear map

$$\epsilon : M \longrightarrow Hom_{\mathbb{Z}^+}(S, D),\ m \mapsto [s \mapsto \mu(sm)].$$

Suppose that $\epsilon(m) = \epsilon(m')$ for some $m, m' \in M$. Then, in particular, $\epsilon(m)(1_S) = \epsilon(m')(1_S)$, i.e. $\mu(m) = \mu(m')$. Since μ is injective, we conclude that $m = m'$. ■

The embedding into an injective R-module (where R is a ring) implies a nice result in the category of R-modules: an R-module P is projective if and only if P is J-projective for every injective R-module J [19, page 411]. For semimodules, so far we have the following implication.

Proposition 2.38 *Let $\gamma : T \longrightarrow S$ be a morphism of semirings and M a left S-semimodule. If ${}_TA$ is ${}_TM$-i-injective, then $Hom_T(S, A)$ is ${}_SM$-i-injective.*

Proof. Let $\iota : K \to M$ be a normal S-monomorphism and $f : K \to Hom_T(S_S, A)$ an S-linear map.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\iota"] \arrow[d, "f"] & M \\ & Hom_T(S_S, A) & \end{tikzcd}$$

Recall the canonical *isomorphism* of commutative monoids $Hom_S(K, Hom_T(S_S, A)) \overset{\theta_{K,A}}{\simeq} Hom_T(K, A)$. Consider the T-linear map $\theta_{K,A}(f) : K \longrightarrow A$.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\iota"] \arrow[d, "\theta_{K,A}(f)"'] & M \arrow[dl, dashed, "h"] \\ & A & \end{tikzcd}$$

Since $\iota : K \to M$ is also a normal T-monomorphism and ${}_TA$ is M-i-injective, there exists a T-linear map $h : M \longrightarrow A$ such that $h \circ \iota = \theta_{K,A}(f)$. Notice that $\theta^{-1}_{M,A}(h) : M \to Hom_T(S_S, A)$ is S-linear and that for all $k \in K$ and every $s \in S$ we have

$$\begin{aligned}
((\theta^{-1}_{M,A}(h) \circ \iota)(k))(s) &= \theta^{-1}_{M,A}(h)(s\iota(k)) &&= h(s\iota(k)) \\
&= (h \circ \iota)(sk) &&= \theta_{K,A}(f)(sk) \\
&= f(sk)(1_S) &&= (sf(k))(1_S) \\
&= f(k)(1_S \cdot s) &&= f(k)(s).
\end{aligned}$$

Hence, $Hom_T(S_S, A)$ is M-i-injective as a left S-semimodule.■

The following result is a combination of Proposition 2.38 and [6, Corollary 3.5].

Corollary 2.39 *Let $\gamma : T \longrightarrow S$ be a morphism of semirings. The functor*

$$Hom_T(S_S, -) : \ _T\mathbf{SM} \longrightarrow \ _S\mathbf{SM}$$

preserves injective, e-injective and i-injective objects.

Lemma 2.40 *Every divisible commutative monoid is $\mathbb{Z}^+$-i-injective.*

Proof. Let D be a divisible commutative monoid, $f : I \to \mathbb{Z}^+$ a normal monomorphism of commutative monoids and $g : I \to D$ a morphism of commutative monoids. Since $f(I)$ is subtractive, $f(I) = k\mathbb{Z}^+$ for some $k \in \mathbb{Z}^+$. Let $i_0 \in I$ be such that $f(i_0) = k$ and notice that i_0 is unique as f is injective. By our choice, D is divisible and so there exists $d \in D$ such that $kd = g(i_0)$. The map

$$h : \mathbb{Z}^+ \to D,\ n \mapsto nd$$

is a well defined morphism of monoids. Moreover, for every $i \in I$, there exsits some $n \in \mathbb{Z}^+$ such that $i = ni_0$, whence $f(i) = f(ni_0) = nf(i_0) = nk$, and so $(h \circ f)(i) = h(nk) = h(n)k = ndk = ng(i_0) = g(ni_0) = g(i)$. It follows that $hf = g$. ■

Definition 2.41 *We say that a left S-semimodule I is c-**injective**, c-e-**injective**, c-i-**injective** if I is M-injective (resp., M-e-injective, M-i-injective) for every cancellative left S-semimodule M.*

Proposition 2.42 *Every divisible commutative monoid is c-i-injective.*

Proof. Let D be a divisible commutative monoid and let $f : M \to N$ be a normal $\mathbb{Z}^+$-monomorphism and $g : M \to J$ a morphism of commutative monoids.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "g"] & N \\ & J & \end{tikzcd}$$

Define

$$\mathcal{S} = \{(A, \alpha) : A \leq_{\mathbb{Z}^+} N,\ M \subseteq A,\ \alpha : A \to J \text{ with } \alpha(m) = g(m)\ \forall\ m \in M\}.$$

Notice that $\mathcal{S}$ is not empty, since $(M, g) \in \mathcal{S}$. Define an order on $\mathcal{S}$ as follows:

$$(A, \alpha) \leq (B, \beta) \Leftrightarrow A \subseteq B \text{ and } \beta(a) = \alpha(a)\ \forall a \in A.$$

Let $((A_\lambda, \alpha_\lambda))_\Lambda$ be a chain in $\mathcal{S}$. Set $\mathbf{A} := \bigcup\limits_{\lambda \in \Lambda} A_\lambda$ and define $\alpha : A \to J$ such that, if $x \in A_\lambda$, then $\alpha(x) = \alpha_\lambda(x)$. Notice that α is well defined, thus the chain has an upper bound $(\mathbf{A}, \alpha)$. By Zorn's Lemma, $\mathcal{S}$ has a maximal element (C, γ).

Claim: If $A \neq N$, then $(A, \alpha) \in \mathcal{S}$ is not maximal.

Let $(A, \alpha) \in \mathcal{S}$ with $A \subsetneqq N$. Choose $b \in N \backslash A$ and set $B := A + \mathbb{Z}^+ b$. Notice that $L := \{r \in \mathbb{Z}^+ |\ rb \in A\}$ is an ideal of $\mathbb{Z}^+$ and

$$\kappa : L \longrightarrow J,\ r \mapsto \alpha(rb)$$

is a morphism of monoids. By Lemma 2.40 there exists a morphism of monoids $\chi : \mathbb{Z}^+ \to J$ such that $\chi(r) = \alpha(rb)\ \forall\ r \in L$. Define

$$\beta : B \to J,\ a + rb \mapsto +\alpha(a) + \chi(r).$$

We claim that β is well defined. Suppose that $a + rb = a' + r'b$ for some $r \in L$ and $a \in A$. Assume, without loss of generality, that $r' > r$, whence $r' = r + \tilde{r}$ for some $\tilde{r} \in \mathbb{Z}^+$. It follows that $a + rb = a' + r'b = a' + rb + \tilde{r}b$, whence $a = a' + \tilde{r}b$

as N is cancellative. It follows that

$$\begin{array}{rcccl} \beta(a'+r'b) & = & \beta((a'+\tilde{r}b)+rb) & = & \alpha(\tilde{r}b+a')+\chi(r) \\ & = & \alpha(a)+\chi(r) & = & \beta(a+rb). \end{array}$$

Thus β is well defined as morphism of monoids with $\beta(a)=\alpha(a)\ \forall\ a\in A$. Thus (A,α) is not maximal in . It follows that there exists a morphism of monoids $h:N\longrightarrow J$ such that (N,h) is maximal in $\mathcal{S}$. Clearly, $h:N\to J$ such that $h\circ f=g$. ■

The following result is, in some sense, a generalization of the fact (mentioned without proof in [18, 17.35]) that any *cancellative semimodule* over semiring can be embedded in a c-injective module. While c-i-injectivity is formally weaker that of c-injectivity, our result works for arbitrary, not necessarily cancellative, semimodules over semirings.

Theorem 2.43 *Every left S-semimodule can be embedded as a subsemimodule of a c-i-injective left S-semimodule.*

Proof. Let M be a left S-semimodule. By Lemma 2.37, M can be embedded as a subtractive subsemimodule of the left S-semimodule $Hom_{\mathbb{Z}^+}(S,D)$ for some divisible commutative monoid D. Let N be a cancellative left S-semimodule; then N is, in particular, a cancellative commutative monoid. By Proposition 2.42, D is an N-i-injective $\mathbb{N}$-semimodule, whence $Hom_{\mathbb{Z}^+}(S,D)$ is N-i-injective by Proposition 2.38.■

Example 2.44 *Let L be a non-zero commutative monoid. By Lemma 2.36, L can* *be embedded subtractively into a divisible commutative monoid M. By Proposition 2.42, M is c-i-injective. However, L cannot be embedded in any injective commutative monoids, since the only injective commutative monoid is $\{0\}$ by Lemma 2.29.*

2.3 Flat Semimodules

The notion of *exactly flat semimodules* was introduced by Abuhlail [4, 3.3] where it was called *normally flat*. The terminology e-flat was first used in [6].

2.45 *Let F_S be a right S-semimodule. Following Abuhlail [4], we say that F_S is a* ***flat*** *right S-semimodule if F is the directed colimit of* finitely presented *projective right S-semimodules.*

Definition 2.46 *Let F_S be a right S-semimodule and ${}_SM$ a left S-semimodule. We say that F is M-e-**flat** if for every short exact sequence $0 \longrightarrow L \overset{\iota}{\longrightarrow} M \overset{\pi}{\longrightarrow} N \longrightarrow 0$ of left S-semimodules, the induced sequence of commutative monoids*

$$0 \longrightarrow F \otimes_S K \overset{F\otimes\iota}{\longrightarrow} F \otimes_S M \overset{F\otimes\pi}{\longrightarrow} F \otimes_S N \longrightarrow 0$$

*is exact. We say that F_S is e-**flat** if F is M-e-flat for every left S-semimodule M.*

2.47 *Let M be a left S-semimodule. A right S-semimodules F is called*

***normally** M-**flat** if for every* subtractive *S-semimodule $L \leq_S M$, we have a* subtractive *submonoid $F \otimes_S L \leq_S F \otimes_S M$;*

*M-i-**flat** if for every* subtractive *S-semimodule $L \leq_S M$, we have a submonoid $F \otimes_S L \leq_S F \otimes_S M$.*

*M-**mono-flat**, if for every S-semimodule $L \leq_S M$, we have a submonoid $F \otimes_S L \leq_S F \otimes_S M$.*

*We say that F_S is **normally flat** (resp., i-**flat**, **mono-flat**) if F is ${}_SM$-e-flat (resp., i-flat, mono-flat) for every left S-semimodule M.*

Remark 2.48 *Let M be a left S-semimodule. It follows directly from the definitions that M-e-flat and M-mono-flat right semimodules are M-i-flat.*

Proposition 2.49 *Let F be a right S-semimodule.*

(1) *F_S is M-e-flat (for some left S-semimodule M) if and only if F_S is normally M-flat.*

(2) *F_S is e-flat if and only if F_S is normally flat.*

Proof. We only need to prove (1).

($\Longrightarrow$) Let $L \leq_S M$ be a subtractive semimodule. Then

$$0 \longrightarrow L \overset{\iota}{\longrightarrow} M \overset{\pi_L}{\longrightarrow} M/L \longrightarrow 0 \tag{2.8}$$

is a short exact sequence of left S-semimodules, where ι is the canonical injection and $\pi_L : M \longrightarrow M/L$ is the canonical projection. Since F_S is M-e-flat, the induced sequence of commutative monoids

$$0 \longrightarrow F \otimes_S L \overset{F\otimes\iota}{\longrightarrow} F \otimes_S M \overset{\pi_L\otimes F}{\longrightarrow} F \otimes_S M/L \longrightarrow 0$$

is exact. In particular, $F \otimes \iota : F \otimes_S L \longrightarrow F \otimes_S M$ is a normal monomorphism. Consequently, F_S is normally M-flat.

($\Longleftarrow$) Let $0 \longrightarrow L \xrightarrow{\iota} M \xrightarrow{\pi} N \longrightarrow 0$ be an exact sequence of left S-semimodules. By our assumption, $0 \longrightarrow F \otimes_S L \xrightarrow{F \otimes \iota} F \otimes_S M$ is a normal monomorphism, whence it follows by Proposition 1.43 (3) that the sequence of commutative monoids

$$0 \longrightarrow F \otimes_S L \xrightarrow{F\otimes\iota} F \otimes_S M \xrightarrow{F\otimes\pi} F \otimes_S N \longrightarrow 0$$

is exact.■

Proposition 2.50 ([4, Theorem 3.6]) *Let S be a semiring. Flat semimodules are e-flat.*

Lemma 2.51 (1) *Let M be a left S-semimodule. Any retract of an M-e-flat right S-semimodule is M-e-flat.*

(2) *Any retract of an e-flat right S-semimodule is e-flat.*

Proof. We only need to prove "1". Let M be a left S-semimodule, $U \leq_S M$ a subtractive subsemimodule, F_S an M-e-flat right S-semimodule and $\widetilde{F}$ a retract of F. Then there exist S-linear maps $\widetilde{F} \xrightarrow{\psi} F \xrightarrow{\theta} \widetilde{F}$ such that $\theta \circ \psi = \mathrm{id}_{\widetilde{F}}$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{F} \otimes_S U \arrow[r, "\mathrm{id}_{\tilde{F}} \otimes_S \iota_U"] \arrow[d, "\psi \otimes_S \mathrm{id}_U"'] & \tilde{F} \otimes_S M \arrow[d, "\psi \otimes_S \mathrm{id}_M"] \\
0 \arrow[r] & F \otimes_S U \arrow[r, "\mathrm{id}_F \otimes_S \iota_U"] \arrow[d, "\theta \otimes_S \mathrm{id}_U"'] & F \otimes_S M \arrow[d, "\theta \otimes_S \mathrm{id}_M"] \\
& \tilde{F} \otimes_S U \arrow[r, "\mathrm{id}_{\tilde{F}} \otimes_S \iota_U"] & \tilde{F} \otimes_S M
\end{tikzcd}$$

Indeed, $(\theta \otimes_S \mathrm{id}_U) \circ (\psi \otimes_S \mathrm{id}_U) = \mathrm{id}_{\widetilde{F} \otimes_S U}$ and $(\theta \otimes_S \mathrm{id}_M) \circ (\psi \otimes_S \mathrm{id}_M) = \mathrm{id}_{\widetilde{F} \otimes_S M}$, *i.e.* $\widetilde{F} \otimes_S U$ is a retract of $F \otimes_S U$ and $\widetilde{F} \otimes_S M$ is a retract of $F \otimes_S M$. Since F_S is M-e-flat, $\mathrm{id}_F \otimes_S \iota_U : F \otimes_S U \longrightarrow F \otimes_S M$ is normal monomorphism. It follows that $\mathrm{id}_{\widetilde{F}} \otimes_S \iota_U$ is injective and indeed normal by Lemma 1.24 "1", *i.e.* $\widetilde{F} \otimes_S U \leq_S \widetilde{F} \otimes_S M$ is a subtractive S-semimodule. Consequently, $\widetilde{F}$ is M-e-flat.■

Proposition 2.52 *Let $\{F_\lambda\}_\Lambda$ be a family of right S-semimodules.*

(1) *Let M be a left S-semimodule. Then $\bigoplus\limits_{\lambda \in \Lambda} F_\lambda$ is M-e-flat if and only if F_λ is M-e-flat for every $\lambda \in \Lambda$.*

(2) *$\bigoplus\limits_{\lambda \in \Lambda} F_\lambda$ is e-flat if and only if F_λ is e-flat for every $\lambda \in \Lambda$.*

Proof. We only need to prove "1".

($\Longrightarrow$) For every $\lambda \in \Lambda$, F_λ is a retract of $\bigoplus\limits_{\lambda \in \Lambda} F_\lambda$, whence M-e-flat by Lemma 2.51.

($\Longleftarrow$) Let $F := \bigoplus\limits_{\lambda \in \Lambda} F_\lambda$ and consider the projections $\pi_\lambda : F \longrightarrow F_\lambda$, $(f_\lambda)_\Lambda \mapsto f_\lambda$ for $\lambda \in \Lambda$. Let $U \leq_S M$ be a subtractive S-subsemimodule. Assume that F_λ is M-e-flat for every $\lambda \in \Lambda$. Then $F_\lambda \otimes_S U \leq_S F_\lambda \otimes_S M$ is a subtractive subsemimodule for every $\lambda \in \Lambda$, whence $\bigoplus\limits_{\lambda \in \Lambda}(F_\lambda \otimes_S U) \leq_S \bigoplus\limits_{\lambda \in \Lambda}(F_\lambda \otimes_S M)$ is a subtractive subsemimodule by Lemma 1.25 (1).

Since

$$\bigoplus_{\lambda \in \Lambda}(F_\lambda \otimes_S U) \simeq \bigoplus_{\lambda \in \Lambda} F_\lambda \otimes_S U \text{ and } \bigoplus_{\lambda \in \Lambda}(F_\lambda \otimes_S M) \simeq \bigoplus_{\lambda \in \Lambda} F_\lambda \otimes_S M,$$

we conclude that $\bigoplus\limits_{\lambda\in\Lambda} F_\lambda \otimes_S U \leq_S \bigoplus\limits_{\lambda\in\Lambda} F_\lambda \otimes_S M$ is a subtractive subsemimodule. It follows that $\bigoplus\limits_{\lambda\in\Lambda} F_\lambda$ is normally M-e-flat.■

The proof of the following lemmas are adapted, with appropriate modifications, from classical ring-theoretic proofs which can be found in standard texts (see [34, proposition 2.70, corollary 3.59, proposition 3.60, theorem 4.9]).

Lemma 2.53 *Given a commutative diagram of S-homomorphisms with exact rows*

$$\begin{tikzcd} A' \arrow[r, "i"] \arrow[d, "f"'] & A \arrow[r, "p"] \arrow[d, "g"'] & A'' \arrow[r] \arrow[d, dashed, "h"] & 0 \\ B' \arrow[r, "j"] & B \arrow[r, "q"] & B'' \arrow[r] & 0 \end{tikzcd}$$

there exists a unique S-homomorphism $h : A'' \to B''$ making the augmented diagram commute. Moreover, if f is surjective and g is an isomorphism, then h is an isomorphism.

Proof. Let $a'' \in A''$. Since p is surjective, exists $a \in A$ such that $p(a) = a''$. Consider

$$h : A'' \to B'',\ a'' \mapsto qg(a).$$

We claim that h is well defined. Suppose that $p(u) = a'' = p(a)$. Since p is k-normal, $u + i(a'_1) = a + i(a'_2)$ for some $a'_1, a'_2 \in A'$. it follows that $(q \circ g \circ i)(a'_1) = (q \circ j \circ f)(a'_1) = 0 = (q \circ g \circ i)(a'_2)$. So,

$$\begin{aligned} (q\circ g)(u) &= (q\circ g)(u) + 0 &&= (q\circ g)(u) + (q\circ g\circ i)(a'_1) \\ &= (q\circ g)(u + i(a'_1)) &&= (q\circ g)(a + i(a'_2)) \\ &= (q\circ g)(a) + (q\circ g\circ i)(a'_2) &&= (q\circ g)(a), \end{aligned}$$

thus h is well defined and $h \circ p = q \circ g$ by the definition of h. Clearly, h is unique. Since g and q are surjective, $h \circ p = q \circ g$ is surjective, whence h is surjective. Moreover, since f and p are surjective and g is injective, it follows by [1, Lemma 3.2] that h is injective. Consequently, h is an isomorphism.■

Lemma 2.54 *If A is an S-k-flat right S-semimodule and I is a subtractive left ideal of S, then we have a canonical isomorphism of commutative monoids*

$$A \otimes_S I \overset{\theta_A}{\simeq} AI, \text{ where } \theta_A(a \otimes_S i) := ai.$$

Proof. Let $\kappa : I \to S$ be the inclusion and recall the canonical isomorphism $A \otimes_S S \overset{\varphi_A}{\simeq} A$ (Lemma 1.22). Since A_S is S-k-flat, we have a monomorphism

$$\psi : A \otimes_S I \overset{A\otimes\kappa}{\longrightarrow} A \otimes_S S \overset{\varphi_A}{\longrightarrow} A, \; a \otimes_S i \mapsto ai.$$

It is obvious that $\psi(A \otimes_S I) = AI$. Since A_S is S-e-flat, $A \otimes \kappa$ injective. Restricting the codomain of ψ to $A \otimes_S I$, we obtain an isomorphism of left S-semimodules $A \otimes_S I \overset{\theta_A}{\simeq} AI$. ■

Lemma 2.55 *Let F be a right S-semimodule, $K \overset{\iota}{\hookrightarrow} F$ a subtractive S-semimodule and I a subtractive left ideal of S. If K_S, F_S and $A_S := F/K$ are S-e-flat Then*

$$K \cap FI = KI. \tag{2.9}$$

Proof. Let I be a subtractive left ideal of S. By Lemma 1.29, we have a short exact sequence of right S-semimodules

$$0 \to K \xrightarrow{\iota} F \xrightarrow{\varphi} A \to 0.$$

Claim: $\iota \otimes I : K \otimes_S I \longrightarrow F \otimes_S I$ is i-normal.

Since $I \leq_S S$ is subtractive, we have an short exact sequence of left S-semimodules: $0 \longrightarrow I \xrightarrow{\iota_I} S \xrightarrow{\pi_I} S/I \longrightarrow 0$. Since K_S and F_S are e-flat, we have two short exact sequences of commutative monoids:

$$\begin{array}{lll} 0 & \longrightarrow & K \otimes_S I \overset{K\otimes\iota_I}{\longrightarrow} K \otimes_S S \overset{K\otimes\pi_I}{\longrightarrow} K \otimes S/I \longrightarrow 0; \\ 0 & \longrightarrow & F \otimes_S I \overset{F\otimes\iota_I}{\longrightarrow} F \otimes_S S \overset{F\otimes\pi_I}{\longrightarrow} F \otimes S/I \longrightarrow 0. \end{array}$$

In particular, $K \otimes \iota_I : K \otimes_S I \longrightarrow K \otimes_S S$ and $F \otimes \iota_I : F \otimes_S I \longrightarrow F \otimes_S S$ are i-normal. Consider the commutative diagram of commutative monoids

$$\begin{tikzcd} & K \otimes_S I \arrow[r, "\iota\otimes I"] \arrow[d, "K\otimes\iota_I"'] & F \otimes_S I \arrow[d, "1\otimes\iota_I"] \\ 0 \arrow[r] & K \otimes_S \arrow[r, "\iota\otimes S"] & F \otimes_S S \end{tikzcd}$$

and notice that $\iota \otimes S$ is a normal monomorphism since $K \otimes_S S \overset{\theta_K}{\simeq} K$, $F \otimes_S S \overset{\theta_F}{\simeq} F$ and $\theta_F \circ (\iota \otimes S) = \iota \circ \theta_K$. Moreover, $F \otimes \iota_I$ is a normal monomorphism since F_S is S-e-flat and it follows by Lemma 1.24 (1) (c) that $\iota \otimes I$ is i-normal.

By our assumption and Lemma 1.43, the following sequence of commutative monoids

$$K \otimes_S I \xrightarrow{\iota \otimes I} F \otimes_S I \xrightarrow{\varphi \otimes I} A \otimes_S I \to 0$$

is exact. Since F_S is flat, $F \otimes_S I \overset{\theta_F}{\simeq} FI$ by Lemma 2.54 (where θ_F is the canonical monoid morphism). Consider the following diagram of commutative monoids

$$\begin{tikzcd}
K \otimes_S I \arrow[r, "\iota \otimes I"] \arrow[d, "\theta_K"'] & F \otimes_S I \arrow[r, "\varphi \otimes I"] \arrow[d, "\theta_F"'] & A \otimes_S I \arrow[r] \arrow[d, dashed, "\gamma"] & 0 \\
KI \arrow[r, "\iota'"] & FI \arrow[r, "\pi"] & FI/\iota'(KI) \arrow[r] & 0
\end{tikzcd}$$

Notice that the canonical monoid morphism $\theta_K : K \otimes_S I \to KI$ is surjective, and since $\iota \otimes I$ is i-normal, it follows that by Lemma 1.24 that ι' is i-normal. It follows by Lemma 1.29 that the 2nd row is exact.

By Lemma 2.53, there exists a unique monoid isomorphism

$$\gamma : A \otimes_S I \longrightarrow FI/\iota'(KI)$$

such that the diagram is commutative.

Since $\varphi : F \longrightarrow A$ is surjective, $\varphi(FI) = AI$. Consider the restriction $\varphi|_{FI}$: $FI \to AI$ and notice that $Ker(\varphi|_{FI}) = FI \cap K$. Consider

$$\beta : AI \to FI/(FI \cap K),\ ai \mapsto [fi] \text{ where } \varphi(f) = a.$$

Claim I: β is well defined.

Suppose that $\varphi(f) = a = \varphi(f')$ for some $f, f' \in F$. Since φ is k-normal, there exist $k, k' \in K$ such that $f + k = f' + k'$, which implies $fi + ki = f'i + k'i$ where $ki, k'i \in FI \cap K$, that is $[fi] = [f'i]$. So, β well defined as it is well defined on a generating set of AI.

Claim II: $Ker(\beta) = 0$.

Notice fist that $FI \hookrightarrow F$ is subtractive: F_S is S-e-flat and $0 \to I \to S \to S/I \to 0$ is exact, the short sequences of commutative monoids $0 \to F \otimes_S I \to F \otimes_S S \to F \otimes_S S/I \to 0$ is exact while $FI \cong F \otimes_S I$ (by Lemma 2.54) and $F = FS \cong F \otimes_S S$ as F is S-e-flat.

Now, let $\sum\limits_j a_j i_j \in Ker(\beta)$, *i.e.* then $[\sum\limits_j f_j i_j] = [0]$ for some $f_j \in F$ satisfying $\varphi(f_j) = a_j$. Then $\sum\limits_j f_j i_j + z = z'$ for some $z, z' \in FI \cap K$. So, $\sum\limits_j f_j i_j \in FI \cap K$ as $FI \cap K \hookrightarrow F$ is subtractive. Thus, $\sum\limits_j a_j i_j = \sum\limits_j \varphi(f_j) i_j = \varphi(\sum\limits_j f_j i_j) = 0$.

Consider

$$\sigma : FI/KI \xrightarrow{\gamma^{-1}} A \otimes_S I \xrightarrow{\theta_A} AI \xrightarrow{\beta} FI/(FI \cap K), \ [fi]_{KI} \mapsto [fi]_{FI \cap K}.$$

Clearly, $Ker(\sigma) = (FI \cap K)/KI$. Since A is S-e-flat, θ_A is an isomorphism by Lemma 2.54 . Since $\sigma = \beta \circ \theta_A \circ \gamma^{-1}$, it follows that $Ker(\sigma) \simeq Ker(\beta) = 0$, whence $(FI \cap K)/KI = 0$, *i.e.* $FI \cap K = KI$. ■

2.3.1 Von Neumann Regular Rings

Definition 2.56 *A semiring S is a* ***von Neumann regular*** *semiring if for every $a \in S$ there exists some $s \in S$ such that $a = asa$.*

Assuming all left semimodules of a given *commutative* semiring to be (mono-

) flat forces the semiring to be a von Neumann regular *ring* (see [27, Theorem 2.11]).

The assumption that all left S-semimodules of a (left and right) subtractive semiring are S-e-flat is sufficient for S to be a von Neumann semiring.

Theorem 2.57 *Let S be a left and right subtractive semiring. If every right S-semimodule is S-e-flat, then S is a von Neumann regular semiring.*

Proof. Let $a \in S$. By our assumption, S is right subtractive, whence the right S-semimodule $K := aS$ is a subtractive right ideal of S and

$$0 \longrightarrow aS \longrightarrow S \longrightarrow S/aS \to 0$$

is an exact sequence of right S-semimodules by Lemma 1.29. Indeed, $F := S_S$ is (S)-e-flat. By our assumptions, the right S-semimodules aS and S/aS are both S-e-flat and so it follows by Lemma 2.55 that for every *subtractive* left ideal I of S :

$$aS \cap I = aS \cap SI = K \cap FI = KI = (aS)I.$$

By our assumption, S is left subtractive and so the left ideal $I := Sa \leq_S S$ is a subtractive left ideal. Whence

$$aSa = (aS)(Sa) = aS \cap Sa.$$

It follows that $a \in aSa$, *i.e.* exists some $s \in S$ such that $a = asa$. ■

Corollary 2.58 *If S is subtractive commutative semiring such that every S-semimodule is e-flat, then S is a von Neumann regular semiring.*

CHAPTER 3

NOETHERIAN, ARTINIAN AND SEMISIMPLE SEMIRINGS

As before, $(S, +, 0, \cdot, 1)$ is a semiring and, unless otherwise explicitly mentioned, an S-module is a **left** S-semimodule.

3.1 Noetherian and Artinian Semirings

Definition 3.1 *A left S-semimodule M is*

Noetherian *(resp., k-**Noetherian**) if M satisfies ACC on its S-subsemimodules (resp., subtractive S-semimodules).*

Artinian *(resp., k-**Artinian**) if M satisfies DCC on its S-subsemimodules (resp., subtractive S-subsemimodules).*

The corresponding notions for right S-semimodules are defined analogously.

Remark 3.1 *Every direct summand of an S-semimodule is subtractive. Let M be an S-semimodule and L a direct summand of M. Then there exists $N \leq_S M$ such that $M = L \oplus N$. Let $m \in M$ and $l, l' \in L$ be such that $m + l = l'$. Write $m = \tilde{l} + \tilde{n}$ for some $\tilde{l} \in L$ and $\tilde{n} \in N$, whence $m + l = (\tilde{n} + \tilde{l}) + l = \tilde{n} + (\tilde{l} + l) = l'$. Since the sum $N + L$ is direct, $\tilde{n} = 0$, and thus $m = \tilde{l} \in L$.*■

The following result is an easy observation; however, we highlight it as it will be used frequently in the proofs of the main results.

Lemma 3.2 *Let M be an S-semimodule and N be* subtractive *S-subsemimodules of M. If $M = L \oplus K$ and $L \subseteq N$, then $N = L \oplus (K \cap N)$.*

Proof. *Clearly, $L + (K \cap N) \subseteq N$. Let $n \in N$. Since $M = L + K$, there exist $k \in K$ and $l \in L$ such that $n = l + k$. Since $l \in N$ and N is subtractive, we have $k \in N$, whence $n \in L + (K \cap N)$. So, $N = L + (K \cap N)$.*

Now, suppose that $l + k = l' + k'$ for some $l, l' \in L$ and $k, k' \in K \cap N$. Since the sum $L + K$ is direct, $l = l'$ and $k = k'$.■

Example 3.3 *Let $S := M_2(\mathbb{R}^+)$. Consider the left ideals E_1 and E_2 defined in 2.2.1 and the left ideal*

$$N_{\geq 1} := \left\{ \begin{bmatrix} a & c \\ b & b \end{bmatrix} \mid a \leq c, b \leq d, a, b, c, d \in \mathbb{R}^+ \right\}.$$

Then we have $N_{\geq 1} \cap (E_1 \oplus E_2) = N_{\geq 1} \cap S = N_{\geq 1}$, while $N_{\geq 1} \cap E1 = \{0\}$ and

$N_{\geq 1} \cap E_2 = E_2$. *In particular,*

$$N_{\geq 1} \cap (E_1 \oplus E_2) \neq (N_{\geq 1} \cap E1) \oplus (N_{\geq 1} \cap E_2).$$

Notice that $N_{\geq 1}$ *is not subtractive, whence the condition that* N *is a subtractive subsemimodule of* M *in Lemma 3.2 cannot be dropped.*

Definition 3.2 *Let* S *be a semiring,* M *be a left* S*-semimodule, and* $N \leq_S M$ *an* S*-subsemimodule. A* subtractive *left* S*-subsemimodule* $L \leq_S M$ *is a* ***maximal subtractive*** *subsemimodule of* N *if* $L \subsetneqq N$ *and if* L' *is a subtractive subsemimodule of* M *with* $L \subseteq L' \subseteq N$*, then* $L = L'$ *or* $L' = N$.

Lemma 3.4 *If* M *is a k-Noetherian left* S*-semimodule, then every non-zero subsemimodule of* M *contains a maximal subtractive* S*-subsemimodule.*

Proof. *Let* $N \leq_S S$ *be a non-zero subsemimodule and consider*

$$\mathcal{I} := \{L \lneqq_S N | \ L \text{ is a subtractive subsemimodule of } M\}.$$

Notice that $L_0 := \{0_M\} \in \mathcal{I}$. *If* L_0 *is a maximal subtractive subsemimodule of* N*, then we are done. Otherwise, there exists* $L_1 \in \mathcal{I}$ *such that* $L_0 \subsetneqq L_1$. *If* L_1 *is a maximal subtractive subsemimodule of* M*, we are done. Otherwise, there exists* $L_2 \in \mathcal{I}$ *such that* $L_1 \subsetneqq L_2$. *If no such maximal subsemimodule of* M *exists, we obtain a non-terminating strictly ascending chain*

$$L_0 \subsetneqq L_1 \subsetneqq L_2 \subsetneqq \cdots \subsetneqq L_k$$

of subsemimodules of N which are subtractive subsemimodules of M, absurd since M is k-Noetherian.■

Definition 3.3 *The semiring S is* ***left Noetherian*** *(resp.,* ***left k-Noetherian****) if ${}_SS$ is Noetherian (resp., left k-Noetherian), equivalently every ascending chain condition of left (resp., subtractive left) ideals of S terminates;*

left Artinian *(resp., left k-Artinian) if ${}_SS$ is Artinian (resp., left k-Artinian), equivalently every descending chain of left (resp., subtractive left) ideals of S terminates.*

The ***right*** *(k-)****Noetherian*** *and* ***right (k-)Artinian semirings*** *are defined analogously. A semiring which is both left and right (k-)Noetherian is a (k-)****Noetherian****, and a semiring which is both left and right (k-)Artinian is (k-)****Artinian****.*

Example 3.5 *([13]) The semiring $\mathbb{Z}^+$ is Noetherian but not Artinian. Setting $I_k := \{0, k, k+1, k+2, \cdots\}$ yields the strictly descending* non-terminating *chain of ideals of $\mathbb{Z}^+$:*

$$I_1 \supsetneqq I_2 \supsetneqq \cdots \supsetneqq I_k \supsetneqq I_{k+1} \supsetneqq \cdots$$

We provide an example of a semiring which is left k-Artinian and left k-Noetherian but neither left Artinian (nor left Noetherian):

Example 3.6 *Let $S = M_2(\mathbb{R}^+)$. The only subtractive left ideals of S are 0, S*

and

$$
\begin{aligned}
E_2 &= \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\}, \\
N_r &= \left\{ \begin{bmatrix} a & ra \\ b & rb \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\}, \ r \in \mathbb{R}^+.
\end{aligned}
$$

Notice that for $r \neq s$, the ideals N_r, N_s are not comparable. Thus, the longest ascending chain of subtractive left ideals of S is $0 \subsetneq N \subsetneq S$ with $N = E_2$ or $N = N_r$ for some $r \in \mathbb{R}^+$. Whence S is left k-Artinian. Similarly, S is left k-Noetherian.

On the other hand, for every $r \in \mathbb{R}^+$ we have a left ideal of S given by

$$
N_{\geq r} = \left\{ \begin{bmatrix} a & p \\ b & q \end{bmatrix} : p \geq ra, \ q \geq rb, \ a, b, p, q \in \mathbb{R}^+ \right\}.
$$

Thus, we have an infinite strictly descending chain of left ideal that does not terminate

$$
N_1 \supsetneqq N_{\geq 2} \supsetneqq N_{\geq 3} \supsetneqq \cdots \supsetneqq N_{\geq m} \supsetneqq N_{\geq m+1} \supsetneqq \cdots
$$

as well as an infinite ascending chain of left ideals that does not terminate

$$
N_{\geq 1} \subsetneqq N_{\geq \frac{1}{2}} \subsetneqq N_{\geq \frac{1}{3}} \subsetneqq \cdots \subsetneqq N_{\geq \frac{1}{m}} \subsetneqq N_{\frac{1}{m+1}} \subsetneqq \cdots
$$

The following additional example of a k-Noetherian semiring that is not Noetherian was communicated to Abuhlail by T. Nam:

Example 3.7 *The semiring $\mathbb{R}^+[x]$ is k-Noetherian but not Noetherian.*

Proof. The semiring $\mathbf{B}[x]$, where B is the Boolean semifield, is not Noetherian . The surjective morphism of semirings

$$f : \mathbb{R}^+ \longrightarrow \mathbf{B},\ 0 \neq r \mapsto 1 \text{ and } 0 \mapsto 0$$

induces a surjective morphism of semirings $\mathbb{R}^+[x] \longrightarrow \mathbf{B}[x]$, whence $\mathbb{R}^+[x]$ is not Noetherian.

We do not know whether k-Artinian semirings are k-Noetherian. However, we have the following interesting result.

Lemma 3.8 *A left S-semimodule M satisfies ACC on direct summands if and only if M satisfies DCC on direct summands.*

Proof. ($\Longrightarrow$) Assume that M satisfies the Descending Chain Condition on direct summands. Let

$$L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots \subseteq L_i \subseteq L_{i+1} \tag{3.1}$$

be an ascending chain of direct summand of M. Then for every $i \in \mathbb{N}$, there exists an S-subsemimodule $N_i \leq_S M$ such that $M = L_i \oplus N_i$; in particular $M = L_1 \oplus N_1$. Since $L_1 \subseteq L_2$ it follows (taking into consideration Remark 3.1) that

$$L_2 = L_1 \oplus (L_2 \cap N_1),$$

by Lemma 3.2, whence

$$M = L_2 \oplus N_2 = L_1 \oplus (L_2 \cap N_1) \oplus N_2.$$

Since $L_2 \cap N_1 \subseteq N_1$ it follows that

$$N_1 {=} (L_2 \cap N_1) \oplus (N_1 \cap (L_1 \oplus N_2)),$$

by Lemma 3.2, whence

$$M = L_1 \oplus N_1 = L_1 \oplus (L_2 \cap N_1) \oplus (N_1 \cap (L_1 \oplus N_2)).$$

Setting $N_1' := N_1$ and $N_2' := N_1 \cap (L_1 \oplus N_2)$, we have $L_1 \oplus N_1' = M = L_2 \oplus N_2'$ where $N_1' \supseteq N_2'$. Since $M = L_2 \oplus N_2'$ and $L_2 \subseteq L_3$, it follows that

$$L_3 {=} L_2 \oplus (L_3 \cap N_2'),$$

by Lemma 3.2, whence

$$M = L_3 \oplus N_3 = L_2 \oplus (L_3 \cap N_2') \oplus N_3.$$

Since $L_3 \cap N_2' \subseteq N_2'$, we have

$$N_2' {=} (L_3 \cap N_2') \oplus (N_2' \cap (L_2 \oplus N_3)),$$

by Lemma 3.2. Setting $N_3' := N_2' \cap (L_2 \oplus N_3)$, we have $N_2' \supseteq N_3'$ and

$$M = L_2 \oplus N_2' = L_2 \oplus (L_3 \cap N_2') \oplus N_3' = L_3 \oplus N_3'.$$

Continuing this process, we obtain a descending chain

$$N_1' \supseteq N_2' \supseteq \cdots \supseteq N_i' \supseteq N_{i+1}' \supseteq \cdots \tag{3.2}$$

of *direct summands* of M such that $M = L_i \oplus N_i'$ for every $i \in \mathbb{N}$. By our assumption, the descending chain (3.2) of terminates, *i.e.* there exists some $k \in \mathbb{N}$ such that $N_i' = N_k'$ for every $i \geq k$.

Now, for every $i \geq k$, we have $L_k \subseteq L_i$, $M = L_k \oplus N_k'$ and $L_i \cap N_i' = 0$ and so

$$L_i{=}L_k \oplus (L_i \cap N_k') = L_k \oplus (L_i \cap N_i') = L_k,$$

by Lemma 3.2. Thus the ascending chain (3.1) terminates.

($\Longleftarrow$) Assume that M satisfies the Ascending Chain Condition on direct summands. Let

$$N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots \supseteq N_i \supseteq N_{i+1} \supseteq \cdots \tag{3.3}$$

be a descending chain of direct summands of M. For every $i \in \mathbb{N}$, there exists a direct summand $L \leq_S M$ such that $M = N_i \oplus L_i$. Since $N_1 \supseteq N_2$, we have (taking into consideration Remark 3.1):

$$M = N_2 \oplus L_2$$

By Lemma 3.2,

$$N_1{=}N_2 \oplus (N_1 \cap L_2) \text{ and } M = N_1 \oplus L_1 = N_2 \oplus (N_1 \cap L_2) \oplus L_1.$$

Set $K_1 := L_1$ and $K_2 := (N_1 \cap L_2) \oplus L_1$, so that $N_1 \oplus K_1 = M = N_2 \oplus K_2$ and $K_1 \subseteq K_2$.

Now, $N_2 \supseteq N_3$ and $M = N_3 \oplus L_3$, whence

$$N_2{=}N_3 \oplus (N_2 \cap L_3),$$

by Lemma 3.2, and

$$M = N_2 \oplus K_2 = N_3 \oplus (N_2 \cap L_3) \oplus K_2.$$

Set $K_3 := (N_2 \cap L_3) \oplus K_2$, so that $M = N_3 \oplus K_3$ and $K_2 \subseteq K_3$. Continuing this way, we obtain an ascending chain

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_i \subseteq K_{i+1} \subseteq \cdots \tag{3.4}$$

of *direct summands* of ${}_SM$. By our assumption, the ascending chain (3.4) terminates, whence there exists $t \in \mathbb{N}$ such that $K_i = K_t$ for every $i \geq t$. Now, For every $i \geq t$ we have $N_t \supseteq N_i$, $M = N_i \oplus K_i$ and $N_t \cap K_t = 0$ and so

$$N_t = N_i \oplus (N_t \cap K_i) = N_i \oplus (N_t \cap K_t) = N_i,$$

thus the descending chain (3.3) terminates.■

Theorem 3.9 *If every subtractive left ideal of S is a direct summand, then S is left k-Artinian and left k-Noetherian.*

Proof. Assume that every subtractive ideal of S is a direct summand.

Claim I: S is left k-Artinian.

Suppose that

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_i \supseteq I_{i+1} \supseteq \cdots \tag{3.5}$$

is descending chain of left subtractive ideals of S that does not terminate. Assume, without loss of generality, that all inclusions are strict, i.e.

$$I_1 \supsetneqq I_2 \supsetneqq I_3 \supsetneqq \cdots \supsetneqq I_i \supsetneqq I_{i+1} \supsetneqq \cdots .$$

For every $k \in \mathbb{N}$, there exists by our assumption some left ideal $N_k \leq_S S$ such that $S = I_k \oplus N_k$. The left ideals I_k, N_k are non-zero as the chain does not terminate, and are subtractive by Remark 3.1.

Since $I_1 \supseteq I_2$ and $S = I_2 \oplus N_2$, we have

$$I_1 {=} I_2 \oplus (I_1 \cap N_2),$$

by Lemma 3.2. Then $J_1 := I_1 \cap N_2$ is a subtractive left ideal of S, which is non-zero as $I_1 \supsetneqq I_2$, and $I_1 = I_2 \oplus J_1$. Since $I_2 \supseteq I_3$ and $S = I_3 \oplus N_3$, we have

$$I_2 {=} I_3 \oplus (I_2 \cap N_3),$$

by Lemma 3.2. Then $I_2 \cap N_3$ is a subtractive left ideal of S, which is non-zero as $I_2 \supsetneqq I_3$, and $I_1 = I_2 \oplus J_1 = I_3 \oplus J_2 \oplus J_1$. Continuing this process, we obtain at the kth step, a non-zero subtractive left ideal $J_k \leq_S S$ such that

$$I_k = I_{k+1} \oplus J_k \text{ and } I_1 = I_{k+1} \oplus J_k \oplus \cdots \oplus J_1.$$

For each $i \in \mathbb{N}$ and setting $J_i' := J_1 \oplus \cdots \oplus J_i$, we have $S = J_i' \oplus I_{i+1} \oplus N_1$ whence J_i' is subtractive (by Remark 3.1). One can easily show that $J := \bigcup\limits_{i \in \mathbb{N}} J_i'$ is subtractive.

By our assumption, $S = J \oplus N$ for some left ideal of $N \leq_S S$. Thus $1_S = j + n$ for some $j \in J$ and $n \in N$. Since $j \in J_i'$ for some $i \in \mathbb{N}$, it can be written in a unique way as $j = j_1 + j_2 + ... + j_i$ for some uniquely determined $j_k \in J_k$, $k = 1, 2, ..., i$. Since $J_{i+1}' \subseteq J$, the sum $J_{i+1}' + N$ is direct, whence the sum $J_1 + J_2 + ... + J_i + J_{i+1} + N$ is direct. Setting

$$M := J_1 \oplus ... \oplus J_i \oplus N,$$

this means that the sum $J_{i+1} + M$ is direct. For any $s_{i+1} \in J_{i+1} \backslash \{0\}$, we have

$$s_{i+1} = s_{i+1}1_S = s_{i+1}(j_1 + j_2 + ... + j_i + n) = s_{i+1}j_1 + s_{i+1}j_2 + ... + s_{i+1}j_i + s_{i+1}n$$

where $s_{i+1}j_k \in J_k$ for $k = 1, 2, ..., i$ and $s_{i+1}n \in N$. It follows that $s_{i+1} \in J_{i+1} \cap M = 0$, absurd since $s_{i+1} \neq 0$. So, the descending chain (3.5) terminates.

Claim II: S is left k-Noetherian.

Let

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq ... \subseteq I_i \subseteq I_{i+1} \subseteq \cdots \tag{3.6}$$

be an ascending chain of subtractive left ideals of S. Since every direct summand of ${}_SS$ is subtractive (by Remark 3.1), it follows from the proof of **Claim (1)** that ${}_SS$ satisfies DCC on direct summands, whence ${}_SS$ satisfies ACC on direct summands by Lemma 3.8. Since (3.6) is an ascending chain of subtractive left ideals of S, whence of direct summands of ${}_SS$ (by our assumption), the chain terminates.■

Example 3.10 *Let p be a prime number. Every subtractive ideal of the semiring $S = B(p+1, p)$ is a direct summand, and S is k-Artinian and k-Noetherian.*

Proof. S has no non-trivial subtractive ideal, thus every subtractive left ideal of S is a direct summand. S is k-Artinian and k-Noetherian since it has finitely many elements.■

Example 3.11 *Let $S := \mathbb{B}^{\mathbb{N}}$ with the canonical structure of a semiring induced by that on $\mathbb{B}$. Then S has a subtractive left ideal which is not a direct summand*

and S is neither k-Artinian nor k-Noetherian.

Proof. The subtractive left ideal $\bigoplus\limits_{n\in\mathbb{N}} \mathbb{B}$ is not a direct summand. The ascending chain

$$\mathbb{B} \times \prod_{n\geq 2}\{0\} \subsetneqq \mathbb{B}^2 \times \prod_{n\geq 3}\{0\} \subsetneqq ... \subsetneqq \mathbb{B}^i \times \prod_{n\geq i+1}\{0\} \subsetneqq \mathbb{B}^{i+1} \times \prod_{n\geq i+2}\{0\} \subsetneqq ...$$

and the descending chain

$$\{0\} \times \prod_{n\geq 2}\mathbb{B} \supsetneqq \{0\}^2 \times \prod_{n\geq 3}\mathbb{B} \supsetneqq ... \supsetneqq \{0\}^i \times \prod_{n\geq i+1}\mathbb{B} \supsetneqq \{0\}^{i+1} \times \prod_{n\geq i+2}\mathbb{B} \supsetneqq ...$$

do not terminate, thus S is neither k-Noetherian nor k-Artinian.

Let R be a ring. It is well known that if every direct sum of left injective R-modules is injective, then R is left Noetherian [19, page 407]. In the category of S-semimodules we have the following implication.

Theorem 3.12 *Let S be a semiring in which every left S-semimodule can be embedded into an S-i-injective left S-semimodule. If every direct sum of left S-i-injective S-semimodules is S-i-injective, then the semiring S is left k-Noetherian.*

Proof. Let

$$L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots \subseteq L_i \subseteq L_{i+1} \subseteq \cdots \tag{3.7}$$

be a chain of subtractive left ideals of S and consider the left ideal $L =: \bigcup\limits_{n\in\mathbb{N}} L_n$. It is clear that L is subtractive. By our assumption, there exists for every n an S-i-injective left S-semimodule J_n and an embedding $S/L_n \overset{\iota_n}{\hookrightarrow} J_n$. Consider the

S-linear map

$$\varphi_n : S \xrightarrow{\pi_i} S/L_n \overset{\iota_n}{\hookrightarrow} J_n \text{ and } \varphi : L \to J,\ x \mapsto \sum_{k=1}^{\infty} \varphi_k(x).$$

Notice that φ is well defined as each $x \in L$ belongs to L_n for some $n \in \mathbb{N}$ and so $\varphi_k(x) = 0$ for all $k \geq n$, *i.e.* $\varphi(x) = \sum\limits_{k=1}^{\infty} \varphi_k(x) = \sum\limits_{k=1}^{n-1} \varphi_k(x)$. By our assumption, $J := \bigoplus\limits_{n \in \mathbb{N}} J_n$ is S-i-injective and so there exists an S-linear map $\psi : S \longrightarrow J$ such that $\psi \circ \iota = \varphi$.

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\iota"] \arrow[d, "\varphi"'] & S \arrow[dl, dashed, "\psi"] \\ & J & \end{tikzcd}$$

Let $\psi(1_S) = \sum t_k \in J$. Then $\psi(1_S) \in \bigoplus\limits_{k=1}^{m-1} J_k$ for some m, whence $\psi(x) = \psi(x \cdot 1_S) = x\psi(1_S) \in \bigoplus\limits_{k=1}^{m-1} J_k$ for every $x \in L$. In particular, $\varphi_m(x) = (\pi_m \circ \phi)(x) = 0$ where π_m is the projection on J_m. Thus $x \in L_n$ and $L = L_n$, whence $L_k = L_n$ for all $k \geq n$, i.e. the chain terminates. Consequently, S is k-Noetherian.■

The proof of the following corollary uses the fact that every semimodule over additively idempotent semiring can be embedded into an e-injective semimodule. [6, 4.5]

Corollary 3.13 *If S is an additively idempotent semiring such that every direct sum of left S-i-injective S-semimodules is S-i-injective, then S is left k-Noetherian.*

Theorem 3.14 *If S is a semiring such that every short exact sequence of left ideals $0 \to L \to S \to N \to 0$ is left splitting, then S is a left k-Noetherian.*

Proof. Let

$$N_0 \subsetneqq N_1 \subsetneqq N_2 \subsetneqq \ldots$$

be a non-terminating ascending chain of subtractive left ideals of S. Notice that $N := \bigcup_{i \in \mathbb{N}} N_i$ is a subtractive ideal of S, whence (by assumption) the following short exact sequence is left splitting

$$0 \to N \xrightarrow{\iota} S \xrightarrow{\pi} S/N \to 0.$$

Let $h : S \to N$ be an S-linear map such that $h \circ \iota = id_N$. Then $h(1_S) \in N$, that is $h(1_S) \in N_i$ for some $i \in \mathbb{N}$. If $x \in N_{i+1} \backslash N_i$, then

$$x = (h \circ \iota)(x) = h(x) = h(x1_S) = xh(1_S) \in N_i$$

a contradiction. Hence S is k-Noetherian.■

Corollary 3.15 *If S is a semiring such that every left subtractive ideal is S-i-injective, then S is a left k-Noetherian semiring.*

3.2 Semisimple Semirings

Throughout, $(S, +, 0, \cdot, 1)$ is a semiring and, unless otherwise explicitly mentioned, an S-module is a **left** S-semimodule.

Definition 3.16 *A semiring S is*

left ideal-semisimple *(resp.,* ***right ideal-semisimple****) if S is ideal-semisimple as a left (right) S-semimodule, equivalently S is a* finite *direct sum of ideal-simple left (right) ideals.*

left congruence-semisimple *(resp.,* ***right congruence-semisimple****) if S is congruence-semisimple as a left (right) S-semimodule, equivalently S is a* finite *direct sum of congruence-simple left (right) ideals.*

The following theorem is well known in the category of modules (see [19]).

Theorem 3.17 ([19, page 362, 402, 404]) *Let R be a ring. Then the following assertions are equivalent:*

(1) *Every left (right) R-module is R-injective.*

(2) *Every left (right) R-module is injective.*

(3) *Every left (right) R-module is projective.*

(4) *Every short exact sequence of left (right) R-modules $0 \to L \to M \to N \to 0$ splits.*

(5) *Every left (right) ideal of R is a direct summand.*

(6) *R is left (right) semisimple.*

In 2009, a result to similar Theorem 3.17 was proved for *subtractive* semirings. We add a new characterization using S-e-projective semimodules.

Theorem 3.18 *If the semiring S is left subtractive, then the following assertions are equivalent:*

(1) *Every left S-semimodule is S-e-projective.*

(2) *Every left S-semimodule is S-k-projective.*

(3) *Every short exact sequence $0 \to L \to S \to N \to 0$ of left S-semimodules splits.*

(4) *Every left ideal of S is a direct summand.*

(5) *S is left ideal-semisimple.*

Proof. The equivalences: (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follow from [30, 4.4].

(1) $\Rightarrow$ (2) follows from the fact that every S-e-projective left S-semimodule is S-k-projective.

(4) $\Rightarrow$ (1) This is Lemma 2.10 applied to $M = {}_S S$. ■

For an arbitrary semiring, having every semimodule projective or injective or e-injective forces the ground semiring to be a semisimple *ring*. The following observation is a combination of [22, Theorem 3.1] and [6, 5.3]:

Theorem 3.19 *The the following assertions are equivalent for any semiring S :*

(1) *Every left (right) S-semimodule is projective;*

(2) *Every left (right) S-semimodule is injective;*

(3) *Every left (right) semimodule is e-injective;*

(4) *S is a left (right) semisimple **ring**.*

Our next goal is to find a relationship between the left ideal-semisimplicity of S and having all left S-semimodules S-e-projective.

Definition 3.20 *Let M be a left S-semimodule. A subsemimodule $N \leq_S M$ is a **maximal summand** of M if $N \leq_S^{\oplus} M$ a direct summand of M such that $N \neq M$ and for every direct summand $L \leq_S^{\oplus} M$ with $N \subseteq L \subseteq M$, we have $N = L$ or $L = M$. A direct summand $N \leq_S^{\oplus} M$ is called an **irreducible summand** if $\{0\}$ is a maximal direct summand of N.*

Theorem 3.21 *If ${}_S S$ satisfies the ascending chain condition on direct summands, then $S = S_1 \oplus S_2 \oplus \cdots \oplus S_n$, where S_i is an irreducible summand for every $i \in \{1, \cdots, n\}$.*

Proof. By our assumptions and Lemma 3.8, S satisfies also the descending chain condition on direct summands. If S has no non-trivial direct summand, then 0 is the maximal summand of S, thus S is an irreducible summand. If not, let D_0 be a non-trivial direct summand of S. Then

$$\mathcal{D}_1 := \{D \supsetneqq D_0 |\ D \text{ is a direct summand of } S\}$$

is non-empty as $S \in \mathcal{D}_1$. Suppose that there exists $(D_\lambda)_\Lambda$ a non-terminating descending chain in $\mathcal{D}_1$. Then there exists $\lambda_i \in \Lambda$, $i = 0, 1, 2, \cdots$ such that $D_{\lambda_0} \underset{\neq}{\supseteq} D_{\lambda_1} \underset{\neq}{\supseteq} \cdots$, is a non-terminating strictly descending chain in $\mathcal{D}_1$, contradiction by DCC on direct summands of ${}_SS$. Thus, the descending chain $(D_\lambda)_\Lambda$ terminates and has a lower bound.

Since every descending chain in $\mathcal{D}_1$ has a lower bound, it follows by Zorn's Lemma, that $\mathcal{D}_1$ has minimal element, Say D_1. Hence there is no direct summand between D_0 and D_1, that is D_0 is a maximal summand of D_1.

The set

$$\mathcal{D}_{-1} := \{D \underset{\neq}{\subseteq} D_0 |\ D \text{ is a direct summand of } S\}$$

is non-empty as $0 \in \mathcal{D}_{-1}$. Suppose that there exists $(D_\lambda)_\Lambda$ a non-terminating ascending chain in $\mathcal{D}_{-1}$. Then there exist $\lambda_i \in \Lambda$, $i = 0, 1, \cdots$ such that $D_{\lambda_0} \underset{\neq}{\subseteq} D_{\lambda_1} \underset{\neq}{\subseteq} \cdots$, is a non-terminating ascending chain on $\mathcal{D}_{-1}$, contradiction by ACC on direct summands of ${}_SS$. Thus the ascending chain $(D_\lambda)_\Lambda$ terminates and has an upper bound.

Since every ascending chain on $\mathcal{D}_{-1}$ has an upper bound, it follows by Zorn's Lemma, that $\mathcal{D}_{-1}$ has maximal element say D_{-1}. Hence there is no direct summand between D_{-1} and D_0, that is D_{-1} is a maximal summand of D_0. We proved that every non-trivial direct summand is a maximal summand of a direct summand and has a maximal summand.

Now, let D_0 be a non-trivial direct summand of S. Then there exists D_1, a direct summand of ${}_SS$, such that D_0 is a maximal summand of D_1. If D_1 is non-trivial, then there exists D_2, a direct summand of S, such that D_1 is a maximal summand of D_2. Repeating this process over and over, we obtain an ascending chain

$$D_0 \subsetneqq D_1 \subsetneqq D_2 \subsetneqq \cdots$$

of direct summands of ${}_SS$, , which should terminate. Thus, there exists $n \in \mathbb{N}$ such that

$$D_0 \subsetneqq D_1 \subsetneqq D_2 \subsetneqq \cdots \subsetneqq D_n = S$$

and D_i is maximal summand of D_{i+1} for $i = 0, 1, ..., n-1$. Since D_0 is a non-trivial direct summand of S, D_0 has maximal summand D_{-1}. If D_{-1} is non-trivial, then D_{-1} has maximal summand D_{-2}. By repeating this process over and over we obtain a descending chain

$$D_0 \supsetneqq D_{-1} \supsetneqq D_{-2} \supsetneqq \cdots$$

of direct summands, which should terminate. Thus, there exists $m \in \mathbb{N}$ such that

$$D_0 \supsetneqq D_{-1} \supsetneqq D_{-2} \supsetneqq \cdots \supsetneqq D_{-m} = 0$$

and D_{-i} is maximal summand of D_{-i+1} for $i = 1, 2, ..., m$. Hence

$$0 = D_{-m} \subsetneqq D_{-m+1} \subsetneqq \cdots \subsetneqq D_{-1} \subsetneqq D_0 \subsetneqq D_1 \subsetneqq D_2 \subsetneqq \cdots \subsetneqq D_n = S$$

is an ascending chain of direct summands of S such that D_i is a maximal summand of D_{i+1} for $i = -m, -m+1, ..., 0, 1, ..., n-1$.

For $i = -m, -m+1, \cdots, 0, 1, \cdots, n-1$, write $S = D_i \oplus L_i$. Since $D_i \subsetneqq D_{i+1}$, we have

$$D_{i+1}{=}D_i \oplus (D_{i+1} \cap L_i),$$

by Lemma 3.2, with $D_{i+1} \cap L_i \neq 0$. Consider $K_{i+1} := D_{i+1} \cap L_i$. Then

$$S = D_n = K_{-m+1} \oplus K_{-m+2} \oplus \cdots \oplus K_n.$$

Suppose that there exists $i \in \{-m+1, -m+2, \cdots, n\}$ such that K_i is a reducible summand. In this case, there exists a direct summand K of K_i such that $0 \neq K \subsetneqq K_i$. Write $K_i := K \oplus L$. Then

$$S = D_i \oplus L_i = D_{i-1} \oplus K_i \oplus L_i = D_{i-1} \oplus K \oplus L \oplus L_i,$$

thus $D_{i-1} \oplus K$ is a direct summand of S such that

$$D_{i-1} \subsetneqq D_{i-1} \oplus K \subsetneqq D_i$$

, contradiction by maximality of D_{i-1} as summand of D_i.■

Remark 3.22 *If S is a semiring with $S = \bigoplus_{i \in I} N_i$, where N_i is a non-zero left ideal of S for every $i \in I$, then I is finite. To see this, suppose that I is infinite. Since $1 \in S = \bigoplus_{i \in I} N_i$ we have $1_S = \sum_{j=1}^{k} n_{i_j}$ for some $k \in \mathbb{N}, i_j \in I$ and $n_{i_j} \in N_{i_j}$. Let $i \in I \backslash \{i_1, \cdots, i_k\}$ and $n_i \in N_i \backslash \{0\}$. Then $n_i = n_i \cdot 1_S = n_i \cdot \sum_{j=1}^{k} n_{i_j} = \sum_{j=1}^{k} n_i n_{i_j}$,* *contradicting the uniqueness of the representation of n_i in the direct sum.*$\blacksquare$

Proposition 3.23 *Let S be a semiring such that S/I is S-k-projective for every subtractive ideal I of S. Then*

(1) *${}_SS$ satisfies the ascending chain condition on direct summands.*

(2) *$S = S_1 \oplus S_2 \oplus \cdots \oplus S_n$, where S_i is an irreducible summand for every $i \in \{1, \cdots, n\}$. If moreover, S_i is ideal-simple (resp., congruence-simple) for every $i \in \{1, 2, ..., k\}$, then S is ideal-semisimple (resp., congruence-semisimple).*

Proof. Assume that S/I is S-k-projective for every subtractive ideal I of S.

(1) Suppose, without loss of generality, that there is a strictly ascending chain of direct summands of ${}_SS$:

$$N_1 \subsetneqq N_2 \subsetneqq \cdots \subsetneqq N_i \subsetneqq N_{i+1} \subsetneqq \cdots$$

where, for each $i \in \mathbb{N}$ we have $S = N_i \oplus L_i$ for some left ideal $L_i \leq_S^{\oplus} S$. Since, $N_i \subsetneqq N_{i+1}$, by Lemma 3.2 we have N_{i+1}=$N_i \oplus (N_{i+1} \cap L_i)$ with $N_{i+1} \cap L_i \neq 0$ for each $i \in \mathbb{N}$. Setting $K_{i+1} = N_{i+1} \cap L_i$ and $K_1 = N_1$, we have $N_i = K_1 \oplus ... \oplus K_i$ for every $i \geq 2$. Thus

$$K := \bigoplus_{i \in \mathbb{N}} K_i = \bigcup_{i \in \mathbb{N}} N_i$$

is a left ideal of S. Moreover, K is subtractive as can be easily shown, whence we have an exact sequence of left S-semimodules

$$0 \to K \xrightarrow{\iota} S \xrightarrow{\pi} S/K \longrightarrow 0.$$

Since S/K is S-k-projective, there exists an S-linear map $\varphi : S/K \to S$ such that $\pi \circ \varphi = id_{S/K}$. For every $s \in S$, we have $\pi(\varphi([1])) = (\pi \circ \varphi)([1]) = [1]$. Sine π is k-normal, there exist $k, k' \in K$ such that $1 + k = \varphi([1]) + k'$. Write $k = k_1 + \cdots + k_j$ and $k' = k'_1 + \cdots + k'_l$, where $k_i, k_i \in K_i$ for every i, and let $m := max\{j, l\}$. The $k = k_0 + k_1 + ... + k_m$ and $k' = k'_1 + ... + k'_m$ for some $k_i, k'_i \in K_i$. Recall that for every $i \in \mathbb{N}$ we have

$$S = N_i \oplus L_i = (K_1 \oplus ... \oplus K_{i-1}) \oplus K_i \oplus L_i.$$

For every $i \in \mathbb{N}$, let $\pi_i : S \to K_i$ be the canonical projection on K_i and $e_i := \pi_i(1)$. Then, $e_i = e_i 1$ implies $\pi_j(e_i) = \pi_j(e_i 1) = e_i \pi_j(1) = e_i e_j$ and so $e_i e_j = 0$ for every $i \neq j$ and $e_i e_i = e_i$. Since $k, k' \in N_m$, $\pi_{m+1}(k) = 0 = \pi_{m+1}(k')$. Thus $e_{m+1} = \pi_{m+1}(1 + k) = \pi_{m+1}(\varphi([1]) + k') = \pi_{m+1}(\varphi([1]))$. Since $S = N_{m+1} \oplus L_{m+1} = K_1 \oplus ... \oplus K_m \oplus K_{m+1} \oplus L_{m+1}$, $1 = e_1 + ... + e_m + e_{m+1} + l_{m+1} = (e_1 + ... + e_m + l_{m+1}) + e_{m+1}$ for some $l_{m+1} \in L_{m+1}$, whence $\pi(1) = \pi(e_0 + e_1 + ... + e_m + l_{m+1})$, thus $[1] = [e_1 + ... + e_m + l_{m+1}]$.

Notice that

$$\begin{aligned}
\varphi([e_1 + ... + e_m + l_{m+1}]) &= \varphi(\pi(e_0 + e_1 + ... + e_m + l_{m+1})) \\
&= \varphi(\pi((e_1 + ... + e_m + l_{m+1})1)) \\
&= \varphi((e_1 + ... + e_m + l_{m+1})\pi(1)) \\
&= (e_1 + ... + e_m + l_{m+1})\varphi(\pi(1)) \\
&= (e_0 + e_1 + ... + e_m + l_{m+1})\varphi([1]) \\
&= \pi_{m+1}(\varphi([e_0 + e_1 + ... + e_m + l_{m+1}])) \\
&= \pi_{m+1}((e_0 + e_1 + ... + e_m + l_{m+1})\varphi([1])) \\
&= (e_0 + e_1 + ... + e_m + l_{m+1})\pi_{m+1}(\varphi([1])) \\
&= (e_0 + e_1 + ... + e_m + l_{m+1})e_{m+1} \\
&= l_{m+1}e_{m+1} \\
&= l_{m+1}\pi_{m+1}(1) \\
&= \pi_{m+1}(l_{m+1}) \\
&= 0
\end{aligned}$$

So, $\varphi([e_1 + ... + e_m + l_{m+1}]) = 0$. It follows that $[1] = [e_0 + e_1 + ... + e_m + l_{m+1}]$ while $\varphi([e_0 + e_1 + ... + e_m + l_{m+1}]) \neq \varphi([1])$, a contradiction. Hence, every ascending chain of direct summands of S terminates, i.e. S satisfies ascending chain condition on direct summands.

(2) By (1), the assumptions of Theorem 3.21 are satisfied, whence

$$S = S_1 \oplus \cdots \oplus S_n$$

where S_i is an irreducible summand for every $i \in \{1, \cdots, n\}$. If moreover, S_i is ideal-simple (resp., congruence-simple) for every $i \in \{1, \cdots, n\}$, then S is the direct sum of ideal-simple (resp. congruence-simple) left ideals, whence ideal-semisimple (resp. congruence-semisimple).$\blacksquare$

The following result is a combination of Lemma 2.10 and Proposition 3.23.

Corollary 3.24 *If S is a semiring such that every subtractive left ideal is a direct summand, then $S = S_1 \oplus \cdots \oplus S_n$, where S_i is an irreducible summand for every $i \in \{1, \cdots, n\}$. If moreover, S_i is ideal-simple (resp., congruence-simple) for every $i \in \{1, \cdots, n\}$, then S is ideal-semisimple (resp., congruence-semisimple).*

3.25 *Let N be a left S-semimodule. Consider the conditions:*

C1 : *Every subtractive S-subsemimodule $M \leq_S N$ is a direct summand.*

C2 : *For every subtractive S-subsemimodule $M \leq_S N$ and every maximal subtractive S-subsemimodule $L \leq_S M$, the left S-semimodule M/L is left ideal-simple.*

C2′: *For every subtractive subsemimodule $M \leq_S N$ and every maximal subtractive S-subsemimodule $L \leq_S M$, the left S-semimodule M/L is congruence-simple.*

Remark 3.26 *The conditions* **C1** *and* **C2** *(and* **C2′***) are independent:*

(1) *$B(3,2)$ satisfies* **C1** *but neither* **C2** *nor* **C2′**.

(2) *$B(3,1)$ satisfies* **C2** *but not* **C1**.

(3) *$\mathbb{B}^{\mathbb{N}}$ satisfies* **C2′** *but not* **C1**.

(4) $\mathbb{R}^+$ *satisfies* **C2** *but not* **C2′**. *By Example 1.13,* $\mathbb{R}^+$ *is ideal-simple but not congruence-simple. Since* $\mathbb{R}^+$ *is ideal-simple, it has no non-trivial ideal,* $\{0\}$ *is the maximal subtractive subsemimodule of* $\mathbb{R}^+$*, and* $\mathbb{R}^+/\{0\} \simeq \mathbb{R}^+$ *is ideal-simple. Hence* $\mathbb{R}^+$ *satisfies* **C2**. *However,* $\mathbb{R}^+/\{0\} \simeq \mathbb{R}^+$ *is not congruence-simple, thus* $\mathbb{R}^+$ *does not satisfy* **C2′**.

(5) *Let* $(M, +, 0)$ *be a finite lattice which is not distributive.* $\mathbf{E}_M$*, the endomorphism semiring of* M*, satisfies* **C2′** *but not* **C2**. *By Example 1.12,* $\mathbf{E}_M$ *is left congruence-simple but not left ideal-simple. Since* $\mathbf{E}_M$ *is left congruence-simple, it has no non-trivial subtractive left ideals,* $\{0\}$ *is the maximal subtractive ideal of* $\mathbf{E}_M$ *and* $\mathbf{E}_M/\{0\} = \mathbf{E}_M$ *is left congruence-simple. Hence,* $\mathbf{E}_M$ *satisfies* **C2′**. *However,* $\mathbf{E}_M/\{0\} = \mathbf{E}_M$ *is not ideal-simple, thus* $\mathbf{E}_M$ *does not satisfy* **C2**.

The converse of Corollary 3.24 is satisfied when the semiring S is commutative. To achieve this, we first prove the following technical result.

Lemma 3.27 *Let* S *is be a* commutative *ideal-semisimple (congruence-semisimple) semiring and write* $S = S_1 \oplus S_2 \oplus ... \oplus S_k$*, where* S_i *is an ideal-simple ideal of* S *for every* $i \in \{1, , \cdots, k\}$*. Then every subtractive ideal* I *of* S *is a direct summand, and moreover* $I = \bigoplus_{a \in A} S_a$ *for some* $A \subseteq \{1, \cdots, k\}$*.*

Proof. Let I be a subtractive ideal of S and

$$A = \{a \in \{1, \cdots, k\} |\ I \cap S_a \neq \{0\}\}.$$

Let $B := \{1, \cdots, k\}\backslash A$ and write $S_A := \bigoplus_{a\in A} S_a$ and $S_B := \bigoplus_{b\in B} S_b$. For every $a \in A$, the ideal S_a is a (subtractive) ideal of A, thus $I \cap S_a$ is a (subtractive) ideal. Since $0 \neq I \cap S_a \subseteq S_a$ and $I \cap S_a$ is a (subtractive) left ideal, $I \cap S_a = S_a$. Thus $S_A \subseteq I$, and it follows that $I{=}S_A \oplus (S_B \cap I)$, by Lemma 3.2.

Claim: $I \cap S_B = 0$.

Let $1 = e_1 + ... + e_k$ for some $e_i \in S_i$. For every $s_i \in S_i$, $s_i = s_i 1 = s_i(e_1 + e_2 + ... + e_k) = s_ie_1 + s_ie_2 + ... + s_ie_k$. Since $s_ie_j \in S_j$ for every $j \in \{1, , \cdots, k\}$, it follows by the directness of the sum that $s_ie_i = s_i$ and $s_ie_j = 0$ for every $i \neq j$. Therefore $e_is_i = s_i$ and $e_js_i = 0$ for every $i \neq j$. Let $x \in I \cap S_B$, whence $x = \sum_{b\in B} x_b$ where $x_b \in S_b$ for each $b \in B$. For every $\tilde{b} \in B$, we have $x_{\tilde{b}} = \sum_{b\in B} e_{\tilde{b}}x_b = e_{\tilde{b}}x \in I$ as I is an ideal. Thus $x_b = 0$ for every $b \in B$ and $x = 0$.■

Proposition 3.28 *For any semiring S, each of the following conditions implies its successor:*

(1) *Every subtractive ideal of S is a direct summand.*

(2) *Every S-semimodule is S-e-projective.*

(3) *Every S-semimodule is S-k-projective.*

(4) *S/I is S-k-projective for every subtractive ideal I of S.*

(5) *Every short exact sequence $0 \longrightarrow I \longrightarrow S \longrightarrow N \longrightarrow 0$ in ${}_S\mathbf{SM}$ right splits.*

(6) *${}_SS$ satisfies ACC on direct summands.*

(7) *${}_SS$ satisfies DCC on direct summands.*

(8) $S = S_1 \oplus \cdots \oplus S_n$, *where every* S_i *is an irreducible summand.*

Proof. (1) $\Rightarrow$ (2) This follows from Lemma 2.10 applied to $M = {}_SS$. Let M be an irreducible summand of ${}_SS$, i.e. $\{0\}$ is the only maximal direct summand of ${}_SM$. By our assumption, $M \simeq M/0$ is ideal-simple.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Follow directly from the definitions.

(4) $\Longleftrightarrow$ (5) Follows from Proposition 2.9 and Lemma 1.29.

(4) $\Rightarrow$ (6) Follows from Proposition 3.23.

(6) $\Leftrightarrow$ (7) Follows from Lemma 3.8.

(6) $\Rightarrow$ (8) Follows by Theorem 3.21.■

The following result extends the characterizations of ideal-semisimple semirings in Theorem 3.18 to commutative not necessarily subtractive semirings:

Theorem 3.29 *The following assertions are equivalent for a* commutative *semiring* S*:*

(1) *Every subtractive ideal of* S *is a direct summand and* S *satisfies* **C2**.

(2) *Every* S*-semimodule is* S*-e-projective and* S *satisfies* **C2**.

(3) *Every* S*-semimodule is* S*-k-projective and* S *satisfies* **C2**.

(4) S/I *is* S*-k-projective for every subtractive ideal* I *of* S, *and* S *satisfies* **C2**.

(5) *Every short exact sequence* $0 \longrightarrow I \longrightarrow S \longrightarrow N \longrightarrow 0$ *in* ${}_S\mathbf{SM}$ *right splits and* S *satisfies* **C2**.

(6) ${}_SS$ *satisfies ACC on direct summands and* **C2**.

(7) ${}_SS$ *satisfies DCC on direct summands and* **C2**.

(8) $S = S_1 \oplus S_2 \oplus S_3 \oplus ... \oplus S_n$, *where every* S_i *is an irreducible summand and* S *satisfies* **C2**.

(9) S *is ideal-semisimple.*

Proof. By Proposition 3.28, we only need to prove (8) $\Rightarrow$ (9) and (9) $\Rightarrow$ (1).

(8) $\Rightarrow$ (9) Notice that assuming **C2** guarantees that S_i is ideal-simple for $i = 1, \cdots, n$. Whence, S is ideal-semisimple.

(9) $\Rightarrow$ (1) Assume that S is ideal-semisimple and write $S = S_1 \oplus \cdots \oplus S_k$ for some $k \in \mathbb{N}$ with S_i an ideal-simple ideal for $i = 1, \cdots, k$. Let I be a subtractive ideal of S. Since S is commutative, it follows by Lemma 3.27 that $I = \bigoplus\limits_{a \in A} S_a$ for some $A \subseteq \{1, \cdots, k\}$, which implies $S{=}I \oplus \bigoplus\limits_{b \notin A} S_b$, by Lemma 3.2. Hence, I is a direct summand of S.

Claim: ${}_SS$ satisfies **C2**.

Let M be a subtractive ideal of S and L a maximal subtractive subideal of M. Then $M = S_A = \bigoplus\limits_{a \in A} S_a$ and $L = S_C = \bigoplus\limits_{c \in C} S_c$ for some $C \subsetneqq A \subseteq \{1, \cdots, k\}$. Notice that $C \subseteq A$ since $L \subseteq M$. Moreover, $|A \backslash C| = 1$ since $|A \backslash C| = 0$ implies $L = M$ and $|A \backslash C| \geq 2$ implies for $y \in A \backslash C$, $L \subsetneqq S_{C \cup \{y\}} \subsetneqq M$ with $S_{C \cup \{y\}}$ a subtractive ideal of S, contradiction by maximality of L. Write $A \backslash C = \{x\}$ and $B = \{1, 2, ..., k\} \backslash A$. Then $S = S_A \oplus S_B = S_C \oplus S_x \oplus S_B$ where $S_B = \bigoplus\limits_{b \in B} S_b$.

If I is an ideal of S such that $L \subsetneqq I \subseteq M$, then there exists $i \in I \backslash N$. Since $i \in M$, $i = t_C + t_x$ for some $t_C \in S_C, t_x \in S_x$. Notice that $t_x \neq 0$; otherwise, $i = t_C \in N$. Moreover, $0 \neq t_x = e_x t_x = e_x(t_C + t_x) = e_x i \in I$, thus $I \cap S_x \neq 0$, which implies $I \cap S_x = S_x$ as S_x is ideal-simple. Since $S_C \subseteq I$ and $S_x \subseteq I$, we have $M = S_C + S_x \subseteq I$.■

The following result is the "congruence-semisimple" version of Theorem 3.29.

Theorem 3.30 *The following assertions are equivalent for a commutative semiring S:*

(1) *Every subtractive ideal of S is a direct summand and S satisfies* $\mathbf{C2}'$.

(2) *Every S-semimodule is S-e-projective and S satisfies* $\mathbf{C2}'$.

(3) *Every S-semimodule is S-k-projective and S satisfies* $\mathbf{C2}'$.

(4) *S/I is S-k-projective for every subtractive ideal I of S and S satisfies* $\mathbf{C2}'$.

(5) *Every short exact sequence $0 \longrightarrow I \longrightarrow S \longrightarrow N \longrightarrow 0$ right splits and S satisfies* $\mathbf{C2}'$.

(6) *${}_SS$ satisfies ACC on direct summands and S satisfies* $\mathbf{C2}'$.

(7) *${}_SS$ satisfies DCC on direct summands and S satisfies* $\mathbf{C2}'$.

(8) *$S = S_1 \oplus \cdots \oplus S_n$, where every S_i is an irreducible summand and S satisfies* $\mathbf{C2}'$.

(9) *S is congruence-semisimple.*

Proof. We only need to prove (9) ⇒ (1); the proof of the other implications are similar to the proof of the corresponding ones in Theorem 3.29.

Assume that S is congruence-semisimple. With the help of Lemma 3.27, it can be shown, as in the proof of Theorem 3.29, that every subtractive ideal of S is a direct summand.

Claim: S satisfies **C2′**.

Let M, L be subtractive ideals of S with L a maximal subtractive S-subsemimodule of M. Then similarly to the proof of Theorem 3.29, we have $M = S_A := \bigoplus_{a \in A} S_a$, $S = S_A \oplus S_B$ and $L = S_C := \bigoplus_{c \in C} S_c$ where $C \cup \{x\} = A$.

Let ρ be a congruence relation on S such that $\equiv_L \subsetneqq \rho \subseteq \equiv_M$. Consider the congruence relation ρ' on S_x :

$$t_x \rho' t'_x \Leftrightarrow (t_C + t_x + t_B)\rho(t'_C + t'_x + t'_B) \text{ for some } t_C, t'_C \in S_C, t_B, t'_B \in S_B.$$

Claim I: $\rho' = S_x^2$.

Since $\equiv_N \neq \rho$, there exist $s, s' \in S$ such that $s \not\equiv_L s'$ and $s \rho s'$. Write $s = s_C + s_x + s_B$ and $s' = s'_C + s'_x + s'_B$ for some $s_C, s'_C \in S_C, s_x, s'_x \in S_x, s_B, s'_B \in S_B$. Since $s \equiv_M s'$, there exists $m, m' \in M = S_A$ such that $m + s = m' + s'$, that is $(m + s_C + s_x) + s_B = (m' + s'_C + s'_x) + s'_B$ which implies $s_B = s'_B$ as the sum $S_A + S_B$ is direct. Notice that $s_x \neq s'_x$; otherwise, $s'_C + s = s_C + s'$ where $s_C, s'_C \in S_C = L$, a contradiction with $\equiv_L \neq \rho$). Therefore, $s_x \rho' s'_x$ and $s_x \neq s'_x$, which implies $\rho' = S_x^2$ as S_x is congruence-simple.

Claim II: $\rho = \equiv_M$.

Let $s, s' \in S$ be such that $s \equiv_M s'$ and write $s = s_C + s_x + s_B$, $s' = s'_C + s'_x + s'_B$ for some $s_C, s'_C \in S_C, s_x, s'_x \in S_x, s_B, s'_B \in S_B$. Then $s_B = s'_B$. Since $\rho' = S_x^2$, we have $s_x\rho' s'_x$, whence $(t_C+s_x+t_B)\rho(t'_C+s'_x+t'_B)$ for some $t_C, t'_C \in S_C, t_B, t'_B \in S_B$. Thus $e_x(t_C+s_x+t_B)\rho e_x(t'_C+s'_x+t'_B)$, that is $s_x\rho s'_x$. Since $s_C \equiv_L s'_C$, we have $s_x\rho s'_x$, and $s_B = s'_B$, $(s_C+s_x+s_B)\rho(s'_C+s'_x+s'_B)$, that is $s\rho s'$. We conclude that $\rho = \equiv_M$.■

The following result is a combination of Theorems 3.29, 3.30 and 3.9:

Corollary 3.31 *If S is a* commutative *ideal-semisimple (congruence-semisimple) semiring, then S is k-Artinian and k-Noetherian.*

The following examples show that the condition **C2** (resp., **C2′**) cannot be dropped from the assumptions of Theorem 3.29 (resp., Theorem 3.30).

Example 3.32 *Consider the commutative semiring $B(p+1,p)$, where p is an odd prime number.*

(1) *Every subtractive ideal is a direct summand.*

(2) *Every $B(p+1,p)$-semimodule is $B(p+1,p)$-e-projective.*

(3) *$B(p+1,p)$ is not left ideal-semisimple.*

(4) *$B(p+1,p)$ is not congruence-semisimple.*

Proof. Notice that the only ideals of $B(p+1,p)$ are $\{0\}$, $B(p+1,p)$, and $I = \{0,p\}$.

(1) The only subtractive ideals of $B(p+1,p)$ are $\{0\}$ and $B(p+1,p)$, each of which is a direct summand of $B(p+1,p)$.

(2) Since (1) is valid and, it follows by Lemma 2.10, that all $B(p+1,p)$-ideals are $B(p+1,p)$-e-projective.

(3) $B(p+1,p)$ is an irreducible summand, which is not ideal-simple since it contains the ideal I. So, $B(p+1,p)$ is not ideal-semisimple. Thus, $B(p+1,p)$ satisfies **C2** nor **C2$'$**.

(4) $B(p+1,p)$ is not left congruence-simple since $\rho = \{(i,j)|\ i,j \neq 0\}$ is a non trivial congruence relation on $B(p+1,p)$. ■

Example 3.33 *Consider the semiring* $S := B(3,1)$.

(1) $I := \{0,2\}$ *is a subtractive ideal of* $B(3,1)$*, which is not a direct summand of* $B(3,1)$*;*

(2) $B(3,1)$ *is not ideal-semisimple;*

(3) $B(3,1)$ *is not congruence-semisimple.*

Proof. Notice that the only ideals of S are 0, I and S, all of which are subtractive. Moreover, I is the maximal subtractive subsemimodule of S and is clearly not a direct summand of S. Moreover, $\{0_S\}$ is the maximal subtractive ideal of I. Notice that $I/0 \cong \mathbb{B} \cong S/I$ as S-semimodules, whence $I/0$ and S/I are ideal-simple. Thus S is an irreducible summand that is neither ideal-simple (I is a non trivial left ideal of S) nor congruence-simple ($\equiv_I$ is a non trivial congruence relation of S).

Proposition 3.34 *For any semiring S, each of the following conditions implies its successor:*

(1) *Every subtractive left ideal of S is a direct summand.*

(2) *Every left S-semimodule is S-e-injective.*

(3) *Every S-semimodule is S-i-injective.*

(4) *Every subtractive ideal of S is S-i-injective.*

(5) *Every short exact sequence $0 \to L \to S \to N \to 0$ in ${}_S\mathbf{SM}$ is left splitting.*

(6) *S is k-Noetherian.*

(7) *S satisfies ACC on direct summands.*

(8) *S satisfies DCC on direct summands.*

(9) *$S = S_1 \oplus S_2 \oplus S_3 \oplus ... \oplus S_n$, where every S_i is an irreducible summand.*

Proof. (1) $\Rightarrow$ (2) Let J be a left S-semimodule and let $f : M \to S$ be a normal monomorphisms. Whence M is a subtractive ideal of S and f is the canonical embedding. Let $g : M \to J$ be an S-linear map. By the assumption, $S = M \oplus N$ for some left ideal N of S. Let $\pi : S \to M$ be the projection on M (i.e., $\pi \circ f = id_M$). Then $g \circ \pi : S \to J$ is an S-linear map satisfying $(g \circ \pi) \circ f = g$.

Let $h : M \to J$ be another S-linear map satisfying $h \circ f = g$. Write $1_S = e_M + e_N$, where $e_M \in M$ are $e_N \in N$ are uniquely determined, and let $j_0 := h(1_S)$. For every $m \in M$, we have $m = m1_S = m(e_M + e_N) = me_M + me_N$, whence

$me_M = m$ and $me_N = 0$ as the sum $M + N$ is direct. Similarly, $ne_M = 0$ and $ne_N = n$ for every $n \in N$. Define

$$h_1 : S \to J,\ s \mapsto se_N j_0.$$

Then $(h_1 \circ f)(m) = h_1(m) = me_N j_0 = 0$ for every $m \in M$. Moreover, we have

$$\begin{array}{rcccc}(g \circ \pi + h_1)(s) & = & (g \circ \pi)(s) + h_1(s) & = & (g \circ \pi)(se_M + se_N) + h_1(s) \\ & = & g(se_M) + se_N j_0 & = & (h \circ f)(se_M) + se_N j_0 \\ & = & h(se_M) + se_N j_0 & = & se_M j_0 + se_N j_0 \\ & = & s(e_M + e_N) j_0 & = & s j_0 \\ & = & h(s) & = & (h + 0)(s).\end{array}$$

Hence J is S-e-injective.

The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) & (6) $\Rightarrow$ (7) follow from the definitions.

(5) $\Rightarrow$ (6) Follows from Theorem 3.14.

(7) $\Longleftrightarrow$ (8) Follows from Lemma 3.8.

(7) $\Rightarrow$ (9) Follows from Theorem 3.21.■

Theorem 3.35 *Let S be a* commutative *semiring. The following assertions are equivalent.*

(1) *Every subtractive ideal of S is a direct summand and S satisfies* **C2**.

(2) *Every S-semimodule is S-e-injective and S satisfies* **C2**.

(3) *Every S-semimodule is S-i-injective and S satisfies* **C2**.

(4) *Every subtractive ideal of S is S-i-injective and S satisfies* **C2**.

(5) *Every short exact sequence $0 \to L \to S \to N \to 0$ in ${}_S\mathbf{SM}$ is left splitting and S satisfies* **C2**.

(6) *S is k-Noetherian and S satisfies* **C2**.

(7) *S satisfies ACC on direct summands and S satisfies* **C2**.

(8) *S satisfies DCC on direct summands and S satisfies* **C2**.

(9) *$S = S_1 \oplus S_2 \oplus S_3 \oplus ... \oplus S_n$, where every S_i is an irreducible summand and S satisfies* **C2**.

(10) *S is ideal-semisimple.*

Proof. This is a consequence of Proposition 3.34 and the proof of Theorem 3.29.■

The following result is the congruence-semisimple version of Theorem 3.35.

Theorem 3.36 *Let S be a commutative semiring. The following assertions are equivalent.*

(1) *Every subtractive ideal of S is a direct summand and S satisfies* **C2′**.

(2) *Every S-semimodule is S-e-injective and S satisfies* **C2′**.

(3) *Every S-semimodule is S-i-injective and S satisfies* **C2′**.

(4) *Every subtractive ideal of S is S-i-injective and S satisfies* **C2′**.

(5) *Every short exact sequence of S-semimodules $0 \to L \to S \to N \to 0$ is left splitting and S satisfies* $\mathbf{C2'}$.

(6) *S is k-Noetherian and satisfies* $\mathbf{C2'}$.

(7) *S satisfies ACC on direct summands and* $\mathbf{C2'}$.

(8) *S satisfies DCC on direct summands and* $\mathbf{C2'}$.

(9) *$S = S_1 \oplus S_2 \oplus S_3 \oplus ... \oplus S_n$, where every S_i is an irreducible summand and S satisfies* $\mathbf{C2'}$.

(10) *S is congruence-semisimple.*

Combining Theorems 3.29 and 3.35, we obtain the following characterization of commutative ideal-semisimple semirings:

Theorem 3.37 *The following assertions are equivalent for a commutative semiring S:*

(1) *Every subtractive ideal of S is a direct summand and S satisfies* $\mathbf{C2}$.

(2) *Every S-semimodule is S-e-projective (S-k-projective) and S satisfies* $\mathbf{C2}$.

(3) *Every S-semimodule is S-e-injective (S-i-injective) and S satisfies* $\mathbf{C2}$.

(4) *For every subtractive ideal I of S we have: S/I is S-k-projective (I is S-i-injective) and S satisfies* $\mathbf{C2}$.

(5) *Every short exact sequence $0 \longrightarrow I \longrightarrow S \longrightarrow N \longrightarrow 0$ in ${}_S\mathbf{SM}$ right splits (left splits) and S satisfies* $\mathbf{C2}$.

(6) *S is k-Noetherian and satisfies* **C2**.

(7) *${}_SS$ satisfies ACC on the direct summands and* **C2**.

(8) *${}_SS$ satisfies DCC on the direct summands and* **C2**.

(9) *$S = S_1 \oplus S_2 \oplus S_3 \oplus ... \oplus S_n$, where every S_i is an irreducible summand and S satisfies* **C2**.

(10) *S is ideal-semisimple.*

Combining Theorems 3.30 and 3.36, we obtain the following characterization of commutative congruence-semisimple semirings:

Theorem 3.38 *The following assertions are equivalent for a* commutative *semiring S:*

(1) *Every subtractive ideal of S is a direct summand and S satisfies* $\mathbf{C2'}$.

(2) *Every S-semimodule is S-e-projective (S-k-projective) and S satisfies* $\mathbf{C2'}$.

(3) *Every S-semimodule is S-e-injective (S-i-injective) and S satisfies* $\mathbf{C2'}$.

(4) *For every subtractive ideal I of S we have: S/I is S-k-projective (I is S-i-injective) and S satisfies* $\mathbf{C2'}$.

(5) *Every short exact sequence $0 \longrightarrow I \longrightarrow S \longrightarrow N \longrightarrow 0$ in ${}_S\mathbf{SM}$ right splits (left splits) and S satisfies* $\mathbf{C2'}$.

(6) *S is k-Noetherian and satisfies* $\mathbf{C2'}$.

(7) ${}_SS$ *satisfies ACC on the direct summands and* $\mathbf{C2'}$.

(8) ${}_SS$ *satisfies DCC on the direct summands and* $\mathbf{C2'}$.

(9) $S = S_1 \oplus S_2 \oplus S_3 \oplus ... \oplus S_n$, *where every* S_i *is an irreducible summand and* S *satisfies* $\mathbf{C2'}$.

(10) S *is congruence-semisimple.*

The following example shows that the assumption that S is a *commutative* semiring cannot be dropped from Theorem 3.29 or from Theorem 3.35, whence not from our main result: Theorem 3.37.

Example 3.39 *Consider the semiring* $S := M_2(\mathbb{R}^+)$ *from 2.2.1.*

(1) S *is a left ideal-semisimple semiring;*

(2) N_1 *is a subtractive left ideal of* S *which is not a direct summand;*

(3) S/N_1 *is not an* S*-k-projective* S*-semimodule (whence not* S*-e-projective).*

(4) N_1 *is not* S*-e-injective.*

Proof.

(1) The semiring $M_2(\mathbb{R}^+)$ is left ideal-simple since $\mathbb{R}^+$ is a semifield ([31]).

(2) Let K be a left ideal of S such that $S = N_1 + K$. Then $1_S = i + k$ for some $i \in N_1$ and $k \in K$, that is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ b & b \end{bmatrix} + \begin{bmatrix} p & q \\ r & s \end{bmatrix}.$$

Then $p+a=1=s+b$ and $q+a=0=r+b$, whence $a=q=r=b=0$ as $\mathbb{R}^+$ is zerosumfree. Therefore, $i=0$ and $k=1_S$, which implies $K=S$ and $0 \neq N_1 = N_1 \cap K$. Thus, the sum N_1+K is not direct. Consequently, N_1 is a subtractive left ideal of S which is not a direct summand.

(3) Let $\pi : S \to S/N_1$ be the canonical map and id_{S/N_1} be the identity map of S/N_1. Notice that π is a normal epimorphism. Consider

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Suppose that there exists an S-linear map $g : S/N_1 \to S$ such that $\pi g = id_{S/N_1}$. Then $g(\overline{e_1}) \in \pi^{-1}(\overline{e_1})$ and $g(\overline{e_2}) \in \pi^{-1}(\overline{e_2})$. Write $g(\overline{e_1}) = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ for some $p,q,r,s \in \mathbb{R}^+$. Then $\begin{bmatrix} p+k & q+k \\ r+l & s+l \end{bmatrix} = \begin{bmatrix} m+1 & m \\ n & n \end{bmatrix}$ for some $k,l,m,n \in \mathbb{R}^+$, which implies that $r=s$ and $p=q+1$ as $\mathbb{R}^+$ is cancellative. By relabeling, we have $g(\overline{e_1}) = \begin{bmatrix} a+1 & a \\ b & b \end{bmatrix}$ for some $a,b \in \mathbb{R}^+$. Similarly, $g(\overline{e_2}) = \begin{bmatrix} c & c \\ d & d+1 \end{bmatrix}$ for some $c,d \in \mathbb{R}^+$.

Let $x := \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in S$. Then

$$x = \begin{bmatrix} p & 0 \\ r & 0 \end{bmatrix} e_1 + \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} e_2,$$

which implies that

$$g(\overline{x}) = \begin{bmatrix} p & 0 \\ r & 0 \end{bmatrix} g(\overline{e_1}) + \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} g(\overline{e_2}) = \begin{bmatrix} pa+dq+p & pa+dq+q \\ ra+sd+r & ra+sd+s \end{bmatrix}.$$

But $x = \begin{bmatrix} p & 1 \\ r & 0 \end{bmatrix} e_1 + \begin{bmatrix} 0 & q \\ 1 & s \end{bmatrix} e_2$, which implies

$$\begin{aligned} \begin{bmatrix} pa+dq+p & pa+dq+q \\ ra+sd+r & ra+sd+s \end{bmatrix} &= g(\overline{x}) = \begin{bmatrix} p & 1 \\ r & 0 \end{bmatrix} g(\overline{e_1}) + \begin{bmatrix} 0 & q \\ 1 & s \end{bmatrix} g(\overline{e_2}) \\ &= \begin{bmatrix} (pa+dq+p)+b & (pa+dq+q)+b \\ (ra+sd+r)+c & (ra+sd+s)+c \end{bmatrix}, \end{aligned}$$

whence $b = 0 = c$ as $\mathbb{R}^+$ is cancellative. Thus $g(\overline{e_1}) = \begin{bmatrix} a+1 & a \\ 0 & 0 \end{bmatrix}$ for some $a, b \in \mathbb{R}^+$ and $g(\overline{e_2}) = \begin{bmatrix} 0 & 0 \\ d & d+1 \end{bmatrix}$.

Let $y = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$. Notice that $\overline{e_1} = \overline{y}$, whence

$$\begin{bmatrix} a+1 & a \\ 0 & 0 \end{bmatrix} = g(\overline{e_1}) = g(\overline{y}) = \begin{bmatrix} 2a+d+2 & 2a+d+1 \\ 0 & 0 \end{bmatrix},$$

and so $a = 2a + d + 1$. Since $\mathbb{R}^+$ is cancellative, $a + d + 1 = 0$, that is 1 has additive inverse, a contradiction. Hence, there is no such S-linear map g with $\pi g = id_{S/I}$; *i.e.*, S/I is not S-k-projective. Since S/I is not S-k-projective, S/I is not S-e-projective.

(4) This was shown in Example 2.27. ■

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Vitae

Name	:	Rangga Ganzar Noegraha
Nationality	:	Indonesian
Date of Birth	:	09/08/1984
Email	:	*ranggaganzar@yahoo.com*
Permanent Address	:	Jl. Dahlia No. D 19, Perumahan Soreang Indah, Soreang, Bandung, West Java 40314
Academic Background	:	Bachelor Degree in Mathematics from Institut Teknologi Bandung, 2006 Master Degree in Mathematics from Institut Teknologi Bandung, 2010